Via Afrika Mathematics

Via Afrika

Our Teachers. Our Future.

Grade 12 Study Guide

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M. Malan

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Contents

Introduction to Via Afrika Mathematics Grade 12 Study Guide

Woohoo! You made it! If you're reading this it means that you made it through Grade 11, and are now in Grade 12. But I guess you are already well aware of that...

It also means that your teacher was brilliant enough to get the *Via Afrika Mathematics Grade 12 Learner's Book*. This study guide contains summaries of each chapter, and should be used side-by-side with the Learner's Book. It also contains lots of extra questions to help you master the subject matter.

Mathematics – not for spectators

You won't learn anything if you don't involve yourself in the subject-matter actively. Do the maths, feel the maths, and then understand and use the maths.

Understanding the principles

- Listen during class. This study guide is brilliant but it is not enough. Listen to your teacher in class as you may learn a unique or easy way of doing something.
- Study the notation, properly. Incorrect use of notation will be penalised in tests and exams. Pay attention to notation in our worked examples.
- Practise, Practise, Practise, and then Practise some more. You have to practise as much as possible. The more you practise, the more prepared and confident you will feel for exams. This guide contains lots of extra practice opportunities.
- Persevere. We can't all be Einsteins, and even old Albert had difficulties learning some of the very advanced Mathematics necessary to formulate his theories. If you don't understand immediately, work at it and practise with as many problems from this study guide as possible. You will find that topics that seem baffling at first, suddenly make sense.
- Have the proper attitude. You can do it!

The AMA of Mathematics

ABILITY is what you're capable of doing.

MOTIVATION determines what you do.

ATTITUDE determines how well you do it.

"Pure Mathematics is, in its way, the poetry of logical ideas." Albert Einstein

Chapter 1

Number patterns, sequences and series

Overview

Chapter 1 Page 8 Number patterns, sequences and series	**Unit 1 Page 10**	
	Arithmetic sequences and series	• Formula for an arithmetic sequence
	Unit 2 Page 14	
	Geometric sequences and series	• Formula for the n^{th} term of a sequence
	Unit 3 Page 18	
	The sum to n terms (S_n): Sigma notation	• The sum to n terms in an arithmetic sequence • The sum to n terms in a geometric sequence
	Unit 4 Page 28	
	Convergence and sum to infinity	• Convergence

REMEMBER YOUR STUDY APPROACH SHOULD BE:

1. Work through all examples in this chapter of your Learner's Bok.
2. Work through the notes in this chapter of this study guide.
3. Do the exercises at the end of the chapter in the Learner's Book.
4. Do the mixed exercises at the end of this chapter in this study guide.

The only way to STUDY Maths is to DO Maths!

Chapter 1

Number patterns, sequences and series

TABLE 1: SUMMARY OF SEQUENCES AND SERIES			
TYPE	GENERAL TERM: T_n	SUM OF TERMS: S_n	EXAMPLES
Arithmetic Sequence (AS) (also named the linear sequence) Constant **1st difference**	$T_n = a + (n-1)d$ $a = first\ term\ T_1$ $d = constant\ diff.$ $d = T_2 - T_1$ $or\ T_3 - T_2$ etc.	$S_n = \frac{n}{2}[2a + (n-1)d]$ or $S_n = \frac{n}{2}[a + l]$ where $l =$ the last term of the sequence	A) $\underbrace{2\ ;\ 5}_{}\ ;\ \underbrace{8\ ;\ 11}_{}\ ;\ \ldots$ $d =$ +3 +3 +3 $T_n = 2 + (n-1)(3)$ $= 2 + 3n - 3$ $= 3n - 1$ B) 1 ; -4 ; -9 ; … $d =$ -5 -5 $T_n = 1 + (n-1)(-5)$ $= 1 - 5n + 5$ $= -5n + 6$
Geometric Sequence (GS) (also named exponential sequence) Constant ratio	$T_n = ar^{n-1}$ $a = first\ term\ T_1$ $r = constant\ ratio$ $r = \frac{T_2}{T_1}\ or\ \frac{T_3}{T_2}$	$S_n = \frac{a(r^n - 1)}{r - 1}$ Or $S_n = \frac{a(1 - r^n)}{1 - r}$ Or $S_\infty = \frac{a}{1-r}$ Where $-1 < r < 1$ (Converging series)	A) 2 ; -4 ; 8 ; -16 ; … $r =$ x-2 x-2 x-2 $T_n = 2(-2)^{n-1}$ NOT CONVERGING as $r < -1$ B) 3 ; $\frac{3}{2}$; $\frac{3}{4}$; $\frac{3}{8}$; … $r =$ x$\frac{1}{2}$ x$\frac{1}{2}$ x$\frac{1}{2}$ $T_n = 3\left(\frac{1}{2}\right)^{n-1}$ CONVERGING as $-1 < r < 1$
Quadratic Sequence (QS) Constant 2nd difference	$T_n = an^2 + bn + c$ $f =$ 1st difference $s =$ 2nd difference Determine a, b and c using simultaneous equations (see example) Alternatively: $a = s \div 2$ $b = f_1 - 3a$ $c = T_1 - a - b$ where $f_1 =$ first term of first differences		3 ; 8 ; 16 ; 27 ; … f: 5 8 11 s: 3 3 Setup three equations using the first three terms: $T_1 = 3$: $3 = a + b + c$ …(1) $T_2 = 8$: $8 = 4a + 2b + c$ …(2) $T_3 = 16$: $16 = 9a + 3b + c$ …(3) Solving simultaneously leads to: $T_n = \frac{3}{2}n^2 + \frac{1}{2}n + 1$

Number patterns, sequences and series

TYPES OF QUESTIONS YOU CAN EXPECT	STRATEGY TO ANSWER THIS TYPE OF QUESTION	EXAMPLE(S) OF THIS TYPE OF QUESTION
Identify any of the following three types of sequences: Arithmetic (AS), Geometric (GS) and Quadratic (QS)	Determine whether sequence has a • constant 1st difference (AS) • constant ratio (GS) • constant 2nd difference (QS)	See Table 1 above
Determine the formula for the **general term**, T_n, of AS, GS and QS (from Grade 11)	You need to find: • a and d for an AS • a and r for a GS • a, b and c for a QS	See Table 1 above
Determine any **specific term** for a sequence e.g. T_{30}	Substitute the value of n into T_n	See Text Book : Example 1, nr. 1 d and 2 d, p.8 (AS) Example 1, nr. 1 b, 3 b, p.11 (AS) Example 1, nr. 1, p. 15 (GS)
Determine the **number of terms** in a sequence, n, for an AS, GS and QS or the position, n, of a specific given term or when the sum of the series is given	**Substitute** all known variables into the general term to get an equation with $\boldsymbol{n}$ as the only **unknown**. Solve for n. OR **Substitute** all known variables into the S_n-formula to get an equation with $\boldsymbol{n}$ as the only **unknown**. Solve for n. Remember: n must be a natural number (not negative, not a fraction)	See Text Book: Example 1, nr.1 c, p.8 Example 1, nr.1 c, p.11 Example 1, nr. 3, p.15 Example 2, nr.3, p.20 Example 3, nr. 2, p.24
When given two sets of information, make use of simultaneous equations to solve: $\boldsymbol{a}$ **and** $\boldsymbol{d}$ (for an AS) $\boldsymbol{a}$ **and** $\boldsymbol{r}$ (for a GS)	For each set of information given, substitute the values of n and T_n or n and S_n. You then have **2 equations** which you can solve **simultaneously** (by substitution)	See Text Book: Example 1, nr. 3, p.11 (AS) Example 1, nr.2, p.15 (AS) Example 3, nr.3, p.24 (GS)
Determine the value of a variable (x) when given a sequence in terms of x.	For AS use constant difference: $T_3 - T_2 = T_2 - T_1$ For GS use constant ratio: $\frac{T_2}{T_1} = \frac{T_3}{T_2}$	The first three terms of an AS are given by $2x - 4; x - 3; 8 - 2x$ Determine x: $8 - 2x - (x - 3) = x - 3 - (2x - 4)$ $\therefore x = 5$

For a series given in sigma notation: • Determine the **number of terms**	Remember: The "**counter**" indicates the number of terms in the series	$\sum_{k=1}^{n} T_k$ has n terms (counter k runs from 1 to n) $\sum_{k=0}^{n} T_k$ has $(n+1)$ terms (counter runs from 0 to n; so one term extra) $\sum_{k=5}^{n} T_k$ has $(n-4)$ terms (four terms not counted)
• Determine the **value** of the series, in other words, S_n.	Remember the expression next to the $\sum$-sign is the general term, T_n. This will help you to determine a and d or r.	See Text Book: Example 1, p.19
Write a given series in **sigma notation.**	Determine the general term, T_k and number of terms, n and substitute into $\sum_{k=1}^{n} T_k$	Example 1, p.19
Determine the **sum, S_n** , of an AS and a GS (when the number of terms are given or not given)	In some cases you have to **first** determine the **number of terms,** n using T_n. Substitute the values of a, n and d/r into the formula for S_n	See Text Book: Example 2, nr.1 & 2, p.20 Example 3, nr. 1, p.24
Determine whether a GS is converging or not	Converging if $-1 < r < 1$	
Determine S_∞ for a converging GS	Substitute vales of a and r Into formula for S_∞	See Text Book: Example 1, nr. 1, p.29
Determine the value of a variable (x) for which a series will converge, e.g. $(2x+1)+(2x+1)^2+$...	Determine r in terms of x and use $-1 < r < 1$	See Text Book: Example 1, nr. 3, p.29
Apply your knowledge of sequences and series on an applied example (often involving diagram/s)	Generate a sequence of terms from the information given. Identify the type of sequence.	See Text Book: Exercise 5, nr. 6, p.30

Mixed Exercise on sequences and series

1 Consider the following sequence: 5; 9; 13; 17; 21; ...

a Determine the general term.

b Which term is equal to 217?

2 a T_5 of a geometric sequence is 9 and T_9 is 729. Determine the constant ratio.

b Determine T_{10}.

3 The following is an arithmetic sequence: $2x-4$; $5x$; $7x-4$

a Determine the value of x.

b Determine the first 3 terms.

4 Consider the following sequence: 2 ; 7 ; 15 ; 26 ; 40 ; ...

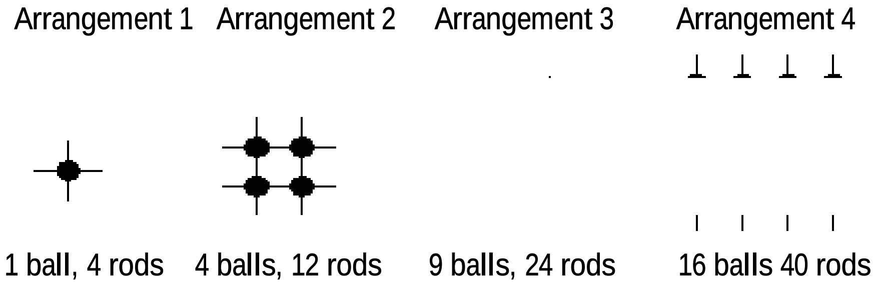
a Determine the general term.

b Which term is equal to 260?

5 How many terms are there in the following sequence?

17 ; 14 ; 11 ; 8 ; ... ; -2785

6 Tom links balls with rods in arrangements as shown below:



a Determine the number of balls in the nth arrangement.

b Determine the number of rods in the nth arrangement.

7 Determine the following:

a $\sum_{k=1}^{30} (8-5k)$ b $\sum_{k=2}^{10} \frac{1}{4}(2)^{k-1}$

8 Write the following in sigma notation: 1+5+9+...+21

9 The 5th term of an arithmetic sequence is zero and the 13th term is equal to 12.

Determine:

a the constant difference and the first term.

b the sum of the first 21 terms.

10 The first two terms of a geometric sequence are: $(x+3)$ and (x^2-9)

a For which value of x is this a converging sequence?

b Calculate the value of x if the sum of the series to infinity is 13.

11 Calculate the value of: $\frac{99+97+95+\cdots+1}{299+297+295+\cdots+201}$

12 $S_n = 3n^2 - 2n$. Determine T_9.

13 The first four terms of a geometric sequence are $7; x; y; 189$.

a Determine the values of x and y.

b If the constant ratio is 3, make use of a suitable formula to determine the number of terms in the sequence that will give a sum of 206 668.

Overview

Chapter 2 Page 36 Functions	**Unit 1 Page 40**	
	The definition of a function	• Relations and functions • Type of relations • Which relations are functions? • Definition of a function? • Function notation
	Unit 2 Page 44	
	The inverse of a function	• The concept of inverses by studying sets of ordered number pairs
	Unit 3 Page 46	
	The inverse of $y = ax + q$	• Graphs of f and f^{-1} on the same set of axes
	Unit 4 Page 48	
	The inverse of the quadratic function $y = ax^2$	• Restricting the domain of the parabola

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TYPES OF RELATIONS BETWEEN TWO VARIABLES			
TYPE	DESCRIPTION	PROPERTIES	TYPICAL EXAMPLES
NON-FUNCTIONS	One-to-many	• One x-value in domain has MORE THAN ONE y-value • Does NOT pass vertical line test	• Inverse of a parabola (See Unit 4)
FUNCTIONS	One-to-one	• Each x-value has a unique y-value • No x- or y-value appear more than once in domain or range • Passes VERTICAL line test	• Straight line graph and its inverse • Hyperbola and its inverse • Exponential graph and its inverse, the logarithmic function
	Many-to- one	• No x-value appears more than once in domain • More than one x-value maps onto the same y-value • Passes VERTICAL line test	• Parabola • Graph of the cubic function • Trigonometric graphs

REVISION OF THE STRAIGHT LINE GRAPH

Standard form: $\boldsymbol{y = mx + c}$

m

- Gradient of line
- Indicates "steepness" and direction of line:

$m > 0\ (+)$ ↗

$m < 0(-)$ ↘

$m = 0$ ↔

- $m = \frac{y_2 - y_1}{x_2 - x_1}$

c

- y-intercept
- Where $x = 0$

PARALLEL AND PERPENDICULAR LINES

Let $y = m_1x + c_1$ and $y = m_2x + c_2$ be two lines.

If the lines are **PARALLEL**, then:

$$m_1 = m_2$$

If the lines are **PERPENDICULAR**, then: $m_1 \times m_2 = -1$

TO DETERMINE THE EQUATION OF A STRAIGHT LINE	
GIVEN:	EXAMPLES
1. Gradient and a point	A line has a gradient of $\frac{1}{2}$ and goes through the point (4;1): $m = \frac{1}{2}$ Substitute point (4;1) into $y = \frac{1}{2}x + c$ $1 = \frac{1}{2}(4) + c$ $c = -1$ $y = \frac{1}{2}x - 1$
2. y-intercept and a point	A line has a y-intercept 3 and goes through the point (-2;1): $c = 3$ Substitute point (-2;1) into $y = mx + 3$ $1 = m(-2) + 3$ $m = 1$ $y = x + 3$
3. Two points on the line	A line goes through the points (4;-3) and (2;1). $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-(-3)}{4-(2)} = 2$ Substitute any one of the two points into $y = 2x + c$ 1=2(2)+c $c = -3$ $y = 2x - 3$
4. A point or y-intercept plus information regarding relationship to another line	a) A line is parallel to the line $y = -x + 3$ and goes through the point (5;-2). **Parallel lines** have same **gradients**; so $m = -1$ Sub (5;-2) into $y = -x + c$ $-2 = -(5) + c$ $c = 3$ b) A line is perpendicular to the line $y = 2x - 1$ and has a y-intercept of 4.

	Perpendicular lines have gradients with a **product of** -1. $m \times 2 = -1 \quad \therefore m = -\frac{1}{2}$ $y = \frac{-1}{2}x + 4$

REVISION OF THE PARABOLA

EQUATION IN STANDARD FORM

$$\boldsymbol{y = ax^2 + bx + c \quad (a \neq 0)}$$

c

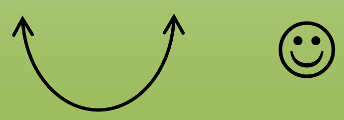
- y-intercept
- Where $x = 0$

a

Indicates shape of parabola

$a > 0\ (+)$

Concave up



Remember:

Positive(+) people smile!

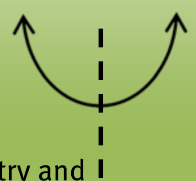
$a < 0(-)$

Concave down

Remember:

Negative (–) people are sad!

b



- Affects the axis of symmetry and turning point (TP)
- Equation of axis of symmetry: $x = -\frac{b}{2a}$
- Coordinates of TP $\left(-\frac{b}{2a}; \frac{4ac-b^2}{4a}\right)$

x-intercepts

- Also called roots/zeroes
- Substitute $y = 0$

EQUATION IN TURNING POINT FORM

$$y = a(x - p)^2 + q \quad (a \neq 0)$$

a

Indicates shape of parabola

$a > 0\ (+)$

Concave up

☺

Remember:

Positive(+) people smile!

$a < 0(-)$

Concave down

☹

Remember:

Negative $(-)$ people are sad!

p and q

- Equation of axis of symmetry $x = p$
- Coordinates of turning point $(p; q)$

Intercepts

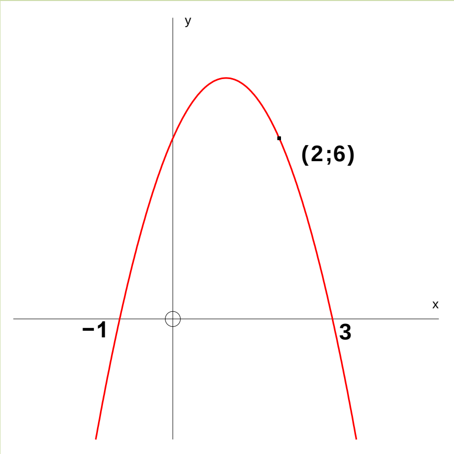
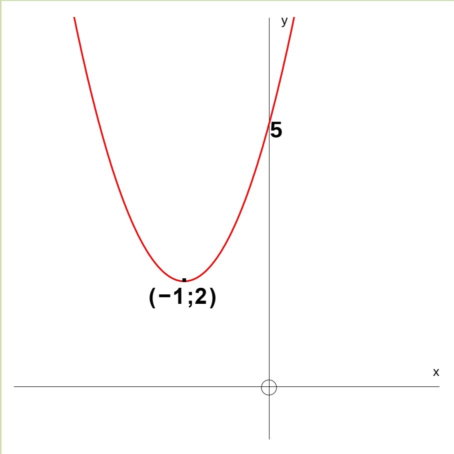
- x-intercepts (make $y = 0$)
- y-intercept (make $x = 0$)

DOMAIN: $x \in R$

RANGE:

$y \in (-\infty; q)$

$y \in (q; \infty)$

DETERMINE THE EQUATION OF A PARABOLA	
GIVEN: 2 ROOTS (x-INTERCEPTS) PLUS 1 POINT	GIVEN: TURNING POINT PLUS 1 POINT
FORM OF EQUATION: $\boldsymbol{y = a(x - x_1)(x - x_2)}$ $\boldsymbol{x_1}$ and $\boldsymbol{x_2}$ are the roots	FORM OF EQUATION: $y = a(x - p)^2 + q$ $(p; q)$ is die turning point of the parabola
EXAMPLE:  $\boldsymbol{x_1 = -1} \qquad \boldsymbol{x_2 = 3}$ $\boldsymbol{y = a(x - x_1)(x - x_2)}$ $\boldsymbol{y = a(x - (-1))(x - 3)}$ $\boldsymbol{y = a(x + 1)(x - 3)}$ Now substitute the other point $\boldsymbol{(2; 6)}$: $\boldsymbol{6 = a(2 + 1)(2 - 3)}$ $\boldsymbol{6 = a(3)(-1)}$ $\boldsymbol{6 = -3a}$ $\boldsymbol{-2 = a}$ $\boldsymbol{y = -2(x + 1)(x - 3)}$ $\boldsymbol{y = -2(x^2 - 2x - 3)}$ $\boldsymbol{y = -2x^2 + 4x + 6}$ (standard form)	EXAMPLE:  $(p; q) = (-1; 2)$ $y = a(x - p)^2 + q$ $y = a(x - (-1))^2 + 2$ $y = a(x + 1)^2 + 2$ Now substitute the point (0;5): $5 = a(0 + 1)^2 + 2$ $5 = a + 2$ $3 = a$ $y = 3(x + 1)^2 + 2$ $y = 3(x^2 + 2x + 1) + 2$ $y = 3x^2 + 6x + 3 + 2$ $y = 3x^2 + 6x + 5$ (standard form)

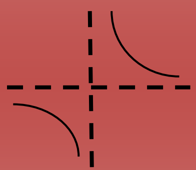
REVISION OF THE HYPERBOLA

$$y = \frac{a}{x-p} + q$$

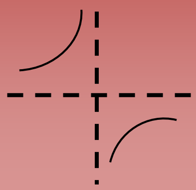
a

Indicates shape of hyperbola (with respect to asymptotes)

$a > 0\ (+)$



$a < 0 (-)$



p

Vertical asymptote

$x = p$

q

Horizontal asymptote

$y = q$

Intercepts

- x-intercept (make $y = 0$)
- y-intercept (make $x = 0$)

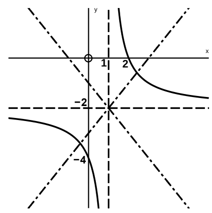
Domain: $x \in R; x \neq p$

Range: $y \in R; y \neq p$

Axes of symmetry (AS)

- Two axes of symmetry
- AS go through intersect of asymptotes $(p; q)$
- Equations: $y = x + k_1$ and $y = -x + k_2$
- Substitute the point $(p; q)$ to calculate k_1 and k_2

EXAMPLE: $y = \frac{2}{x-1} - 2$	
y-intercept: $y = \frac{2}{-1} - 2 = -4$ x- intercept: $0 = \frac{2}{x-1} - 2\ ; x = 2$ Asymptotes: $x = 1\ and\ y = -2$	Axes of symmetry: Substitute $(1; -2)$ into $y = x + k_1\ and\ y = -x + k_2$ $-2 = 1 + k_1\ and\ -2 = -1 + k_2$ $k_1 = -3\ and\ k_2 = -1$ $y = x - 3\ and\ y = -x - 1$



REVISION OF THE EXPONENTIAL GRAPH

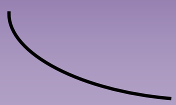
$$y = a^{x-p} + q$$

a

Indicates shape of hyperbola

$a > 1$

$0 < a < 1$



q

- Horizontal asymptote: $y = p$
- Indicates that the graph $y = a^x$ was translated (shifted) **vertically** up/down

  $q > 0$: shifted upwards

  $q < 0$: shifted downwards

p

Indicates that the graph $y = a^x$ was translated (shifted) **horizontally** left/right

$p > 0$: shifted left

$p < 0$: shifted right

Intercepts

- x-intercept (make $y = 0$)
- y-intercept (make $x = 0$)

Domain: $x \in R$

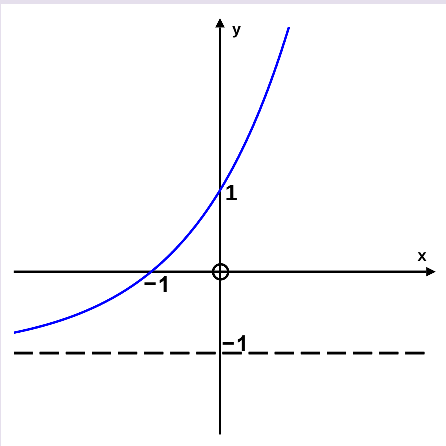
Range: $y \in (q; \infty)$

EXAMPLE: $y = 2^{x+1} - 1$

Asymptote: $y = -1$

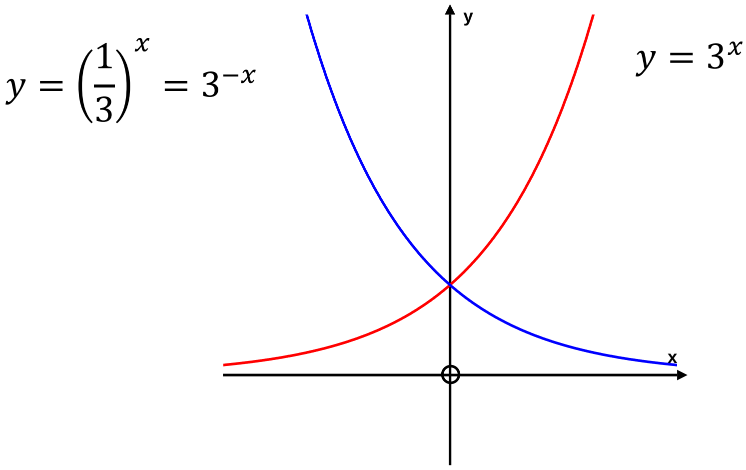
x-intercept ($y = 0$): $2^{x+1} - 1 = 0 \quad \therefore x = -1$

y-intercept: ($x = 0$): $y = 2^{0+1} - 1 = 1$

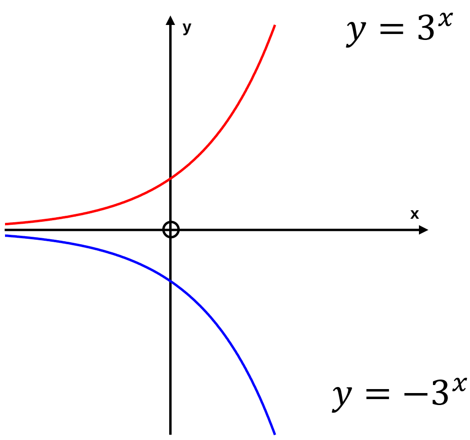


EXAMPLES OF SYMMETRICAL EXPONENTIAL GRAPHS

SYMMETRICAL IN THE y –axis



SYMMETRICAL IN THE x –axis



INTERSECTS OF TWO GRAPHS

To determine the coordinates of the point where two graphs **INTERSECT**:

Use **SIMULTANEOUS EQUATIONS**

EXAMPLE

Determine the coordinates of the points of intersection of $f(x) = 3x + 6$ and $g(x) = -2x^2 + 3x + 14$

Equate the two equations and solve for x:

$$3x + 6 = -2x^2 + 3x + 14$$
$$2x^2 - 8 = 0$$
$$x^2 - 4 = 0$$
$$(x - 2)(x + 2) = 0$$
$$x = 2 \text{ or } x = -2$$

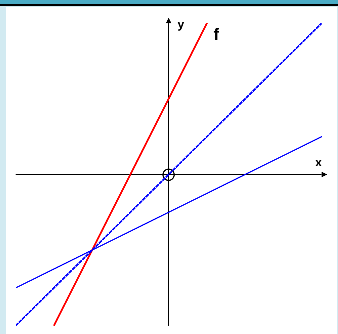
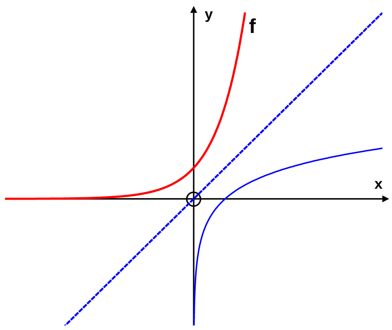
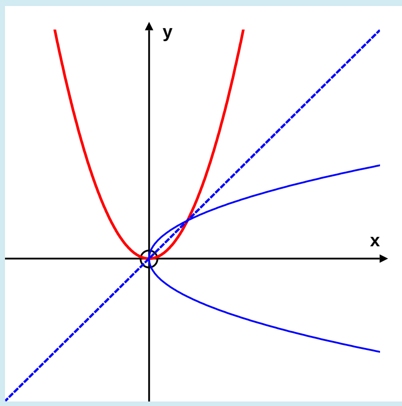
Substitute x-values back into one of equations (choose the easier one):

If $x = 2$ then $y = 3(2) + 6 = 12$
So one point of intersection is $(2; 12)$.

If $x = -2$ then $y = 3(-2) + 6 = 0$
The other point of intersection is $(-2; 0)$ which is also the x-intercept of both graphs.

THE INVERSE OF A FUNCTION

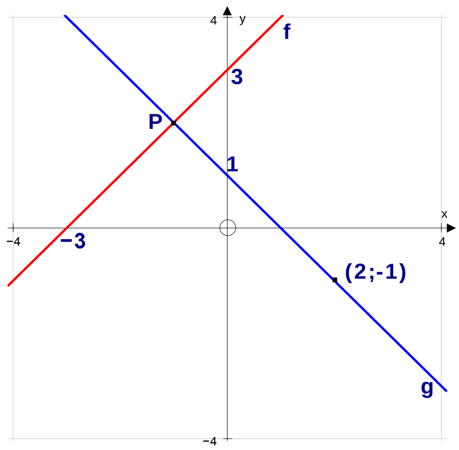
- The inverse of a function, f, is denoted by f^{-1}.
- f^{-1} is a reflection of f in the line $y = x$
- To determine the equation of f^{-1}, swop x and y in the equation of f
- The x-intercept of f is the y-intercept of f^{-1}

FUNCTION f	**INVERSE OF FUNCTION, f^{-1}**	**EXAMPLES**	**DIAGRAM**
Straight line $\boldsymbol{f: y = mx + c}$	Straight line	$f: y = 2x + 3$ Inverse: $2y + 3 = x$ $f^{-1}: y = \frac{1}{2}x - \frac{3}{2}$	
Exponential graph $\boldsymbol{f: y = a^x}$	Logarithmic function $f^{-1}: y = \log_a x$	$f: y = 3^x$ Inverse: $f^{-1}: y = \log_3 x$	
Parabola $\boldsymbol{f: y = ax^2}$	The inverse of a parabola is **NOT A FUNCTION** NB: The DOMAIN of the parabola has to be **RESTRICTED** to $x \geq 0$ or $x \leq 0$ so that f^{-1} is also a function	$f: y = 2x^2$ Inverse: $x = 2y^2$ $y^2 = \frac{1}{2}x$ $f^{-1}: \pm\sqrt{\frac{1}{2}x}$	

Mixed Exercise on Functions

1 Determine the coordinates of the intercept of the following two lines:

$$2x - 3y = 17$$
$$3x - y = 15$$



2 a Determine the equation of line f.

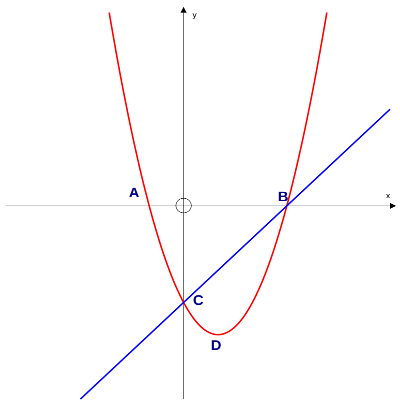
b Determine the equation of line g.

c Determine the co-ordinates of point, P, where the two lines intersect.

d Are these two lines perpendicular? Give a reason for your answer.

e Write down the equation of the line which is parallel to line g with a y-intercept of -2.

3 The diagram shows the graphs of $y = x^2 - 2x - 3$ and $y = mx + c$.



a Determine the lengths of OA, OB and OC.

b Determine the co-ordinates of the turning point D.

c Determine m and c of the straight line.

d Use the graph to determine for which values of k for which the equation $x^2 - 2x + k = 0$ would have only one real root.

4 The diagram shows the graph of $f(x) = -2(x+1)^2 + 8$.

C is the turning point.

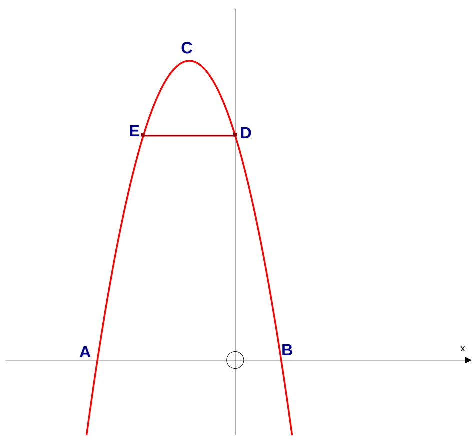
E is the mirror image of the y-intercept of f.

Determine:

a the length of AB.

b the co-ordinates of C.

c the length of DE.



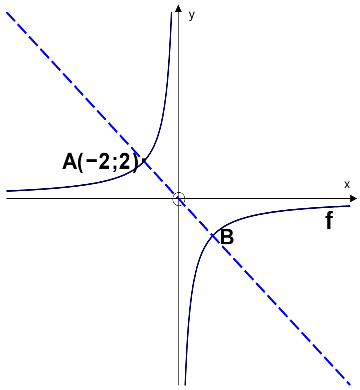
5 Consider the function $g(x) = \left(\frac{1}{2}\right)^x - 2$.

a Make a neat drawing of g. Clearly show the asymptote and intercepts with the axes.

b Determine the domain of g.

c For which values of x would $g(x) \geq 0$.

6 The graph of $f(x) = \frac{a}{x}$; $x \neq 0$ is shown.



$A(-2; 2)$ is a point on the graph where it cuts the line $y = -x$.

a Determine the value of a .

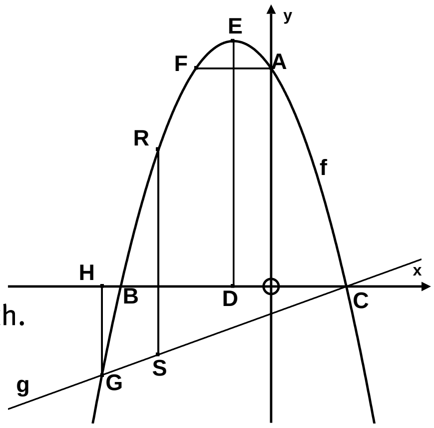
b Write down the coordinates of B.

c Graph f is translated 2 units up and 1 unit right.

d Write down the equation of the new graph.

7 The graphs of the following are shown :
$f(x) = -x^2 - 2x + 8$ and $g(x) = \frac{1}{2}x - 1$

Determine:



a the coordinates for A

b the coordinates for B and C

c the length of CD

d the length of DE which is parallel to the y-axis

e the length of AF which is parallel to the x-axis

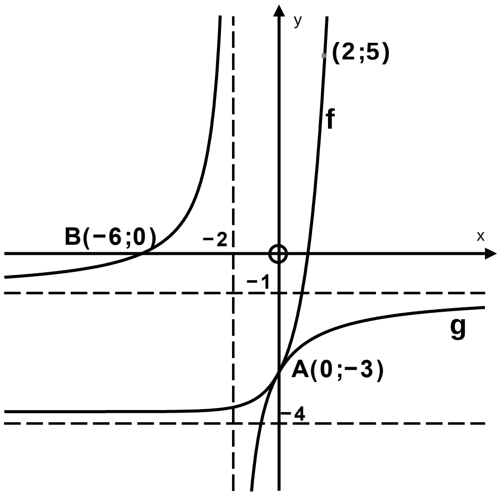
f the length of GH which is parallel to the y-axis

g the x-value for which RS would have a maximum length.

h the maximum length of RS.

i the x-values for which $f(x) - g(x) > 0$.

8 The diagram alongside shows the graphs of the functions of



$$f(x) = b^x + c \quad and \quad g(x) = \frac{a}{x+p} + q$$

a Write down the equation of the asymptote of f.

b Determine the equation of f.

c Write down the equations of the asymptotes of g.

d Determine the equation of g.

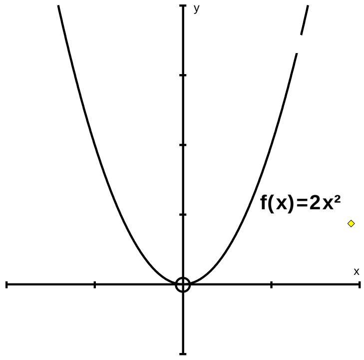
e Determine the equations of the axes of symmetry of g .

f For which values of x is $f(x) > g(x)$?

9 The graph of $f(x) = 2x^2$ is given.

a Determine the equation of f^{-1} in the form $f^{-1}: y = \cdots$

b How can one restrict the domain of f so that f^{-1} will be a function?



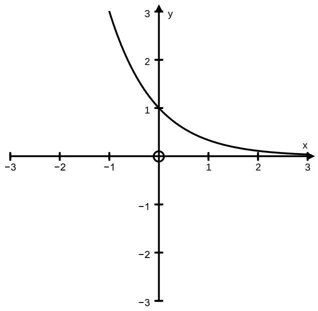
10 The graph of $f(x) = a^x$ is given.
The point A (-1; 3) lies on the graph.

a Determine the equation of f.

b Determine the equation of f^{-1} in the form $f^{-1}: y = \cdots$

c Make a neat drawing of the graph of f^{-1}.

d Determine the domain of f^{-1}.



11 A straight line graph has an x-intercept of -2 and a y-intercept of 3. Write down the **coordinates** of the x- and y-intercepts of f^{-1}.

Chapter 3
Logarithms

Overview

Chapter 3 Page 58 Logarithms	**Unit 1 Page 60**	
	The definition of a logarithm	• Changing exponents to the logarithmic form • Proofs of the logarithmic laws
	Unit 2 Page 64	
	Solve exponential equations using logarithms	• Using logarithms
	Unit 3 Page 66	
	The graph of $y = \log_b x$ where $b > 1$ and $0 < b < 1$	• Inverse of $y = f(x) = 2^x$ • Inverse of the function $y = f(x) = \left(\frac{1}{2}\right)^x$

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Definition of logarithm

If $\log_b x = y$, then $b^y = x$.

EXAMPLES: Converting from one form to another	
Logarithmic form	Exponential form
$\log_3 243 = 5$	$3^5 = 243$
$\log_{0,5} 0,125 = 3$	$0,5^3 = 0,125$
$\log_{10} 1000 = 3$	$10^3 = 1000$
$\log_3 \sqrt{3} = \frac{1}{2}$	$3^{\frac{1}{2}} = \sqrt{3}$

LOGARITHMIC LAW	EXAMPLES
Law 1: $\log_m A.B = \log_m A + \log_m B$	• $\log_k abc = \log_k a + \log_k b + \log_k c$ • $\log_5 25.5 = \log_5 25 + \log_5 5 = 2 + 1 = 3$
Law 2: $\log_m \frac{A}{B} = \log_m A - \log_m B$	• $\log_m \frac{y}{z} = \log_m y - \log_m z$ • $\log_5 \frac{0,2}{25} = \log_5 0,2 - \log_5 25 = \log_5 5^{-1} - \log_5 25 = -1 - 2 = -3$
Law 3: $\log_x P^y = y \log_x P$	• $\log_y a^3 = 3 \log_y a$ • $\log_5 0,04 = \log_5 5^{-2} = -2 \log_5 5 = -2$
Law 4: $\log_b a = \frac{\log a}{\log b}$	• $\log_b a = \frac{\log a}{\log b}$ • $\log_2 5 = \frac{\log 5}{\log 2} = 2,32$

Note that:

- $\log_a a = 1 \ (a \neq 0)$
- $\log_a 1 = 0$
- $\log a = \log_{10} a$

USING LOGARITHMS TO SOLVE EQUATIONS

We know that equations involving exponents can be solved using exponential laws:

$$2^x = 128$$

$$2^x = 2^7 \text{(prime factorise)}$$

$$\therefore x = 7$$

But, what if we cannot use prime factors?

$$2^x = 13$$

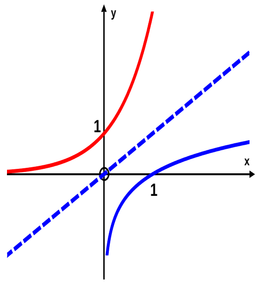
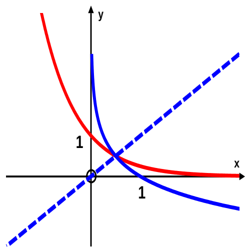
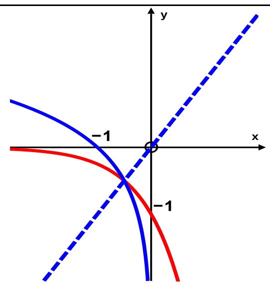
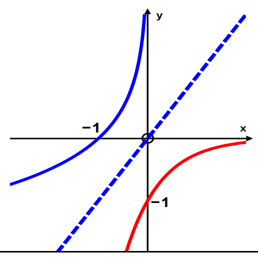
$$\log 2^x = \log 13$$

$$x \log 2 = \log 13$$

$$x = \frac{\log 13}{\log 2} = 3,7$$

THE INVERSE OF THE EXPONENTIAL GRAPH

$f^{-1}: y = \log_a x\,; x > 0$

EXAMPLES		
f(RED GRAPH)	f^{-1}(BLUE GRAPH)	DIAGRAM
$y = 4^x$	$y = \log_4 x$	
$y = \frac{1}{4}^x$	$y = \log_{\frac{1}{4}} x$	
$y = -4^x$	$y = \log_4(-x)$	
$y = -\frac{1}{4}^x$	$y = \log_{\frac{1}{4}}(-x)$	

Mixed Exercise on Logarithms

1 Make use of the definition of the logarithm to solve for x:

a $\log_3 x = 2$

b $\log_{\frac{1}{3}} x = 2$

c $-\log_4 x = 2$

d $\log_5 x = -2$

e $\log\ x^3 = 6$

f $\log_3 81 = x$

g $\log_3 \frac{1}{9} = x$

2 The graph of $f(x) = a^x$ goes through the point $\left(2; \frac{9}{4}\right)$.

a Determine the value of a.

b Determine the equation of f^{-1}.

c Determine the equation of g if f and g are symmetrical in the y-axis.

d Determine the equation of h, the reflection of f^{-1} in x-axis.

3 The function f is given by the graph $f(x) = \log_2 x$.

a Determine the equations of the following graphs:

i g, the reflection of f in the x-axis

ii p, the reflection of f in the y-axis

iii q, the reflection of g in the y-axis

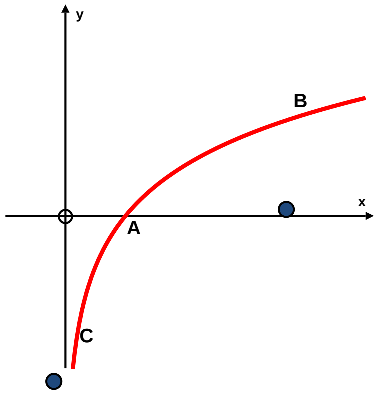
iv f^{-1}, the inverse of f

v g^{-1}, the inverse of g

vi h, the translation of f two units left

b Sketch the graphs of f, f^{-1}, g and g^{-1} on the same system of axes.

c Determine the domain and range of f^{-1} and g^{-1}.



4 The graph of $y = \log_b x$ is shown in the diagram alongside.

a Determine the coordinates of point A.

b How do we know that $b > 1$.

c Determine b if B is the point $\left(8; \frac{3}{2}\right)$.

d Determine the equation of g, the inverse of this graph.

e Determine the value of a if C is the point $(a; -2)$.

Overview

Chapter 4 Page 76 Finance, growth and decay	**Unit 1 Page 78**	
	Future value annuities	• Deriving the future value formula
	Unit 2 Page 82	
	Present value annuities	• Deriving the present value formula
	Unit 3 Page 86	
	Calculating the period	• Finding the value of n
	Unit 4 Page 88	
	Analysing investments and loans	• Outstanding balances on a loan • Sinking fund • Pyramid schemes

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HIRE PURCHASE AGREEMENTS

$$A = P(1 + in)$$

NB: Simple Interest

Example:

Kelvin buys computer equipment on hire purchase for R20 000.
He has to put down 10% deposit and repays the amount monthly over 3 years.
The interest rate is 15% p.a.

Deposit = 10% of R20 000 = R2 000.
He has to repay $A = 18000(1 + 0{,}15 \times \mathbf{3}) = R26\,100$ in total.
36 monthly payments of R26 100$\div$ 36 =R725 each.

INFLATION / INCREASE IN PRICE OR VALUE

NB: Compound Interest

$$A = P(1 + i)^n$$

n= number of **years**

DEPRECIATION

Choose the correct formula!

Straight line method

$$A = P(1 - in)$$

Reducing-balance method

$$A = P(1 - i)^n$$

n= number of **years**

Chapter 4
Finance, growth and decay

NOMINAL AND EFFECTIVE INTEREST RATES

$$(1 + i_{eff}) = \left(1 + \frac{i_{nom}}{m}\right)^{m}$$

NB: m = the **number of times per year** interest is added

Daily: $m = 365$

Monthly: $m = 12$

Quarterly: $m = 4$

Half-yearly (semi-annually): $m = 2$

EXAMPLE:

What is the effective rate if the nominal rate is 18% p.a. compounded quarterly?

In other words:

Which rate **compounded annually** will give me the **<u>same</u>** return as 18% compounded **quarterly**?

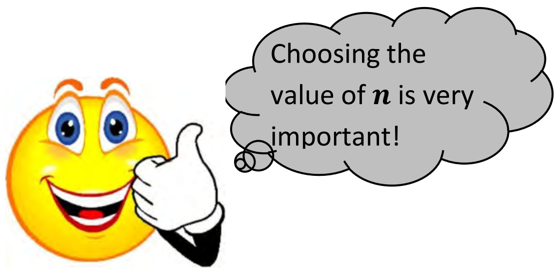
$$i_{eff} = \left(1 + \frac{0{,}18}{4}\right)^{4} - 1$$

$=0{,}1925186\ldots$

Effective rate = 19,25%

FUTURE VALUE ANNUITIES

$$F = \frac{x[(1+i)^n - 1]}{i}$$

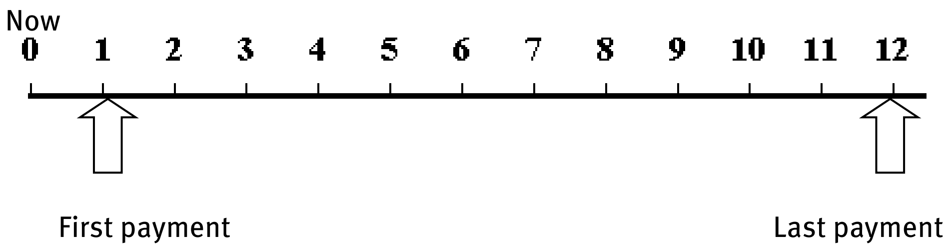


KEY WORDS:

- Regular investments (monthly/quarterly etc.)
- Sinking funds
- Annuity/pension
- Savings plan

<u>Example 1</u>

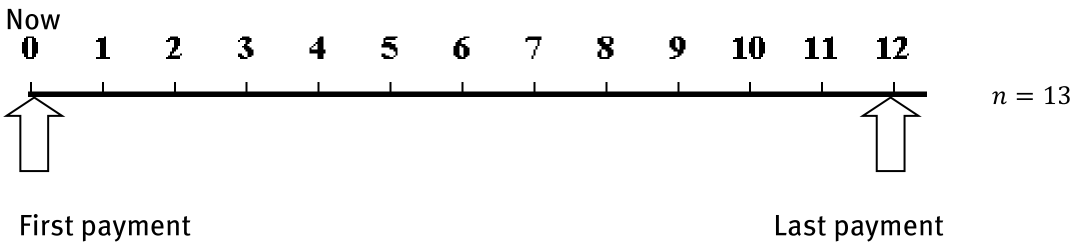
First payment in one month's time. Last payment in one year's time.



$n = 12$

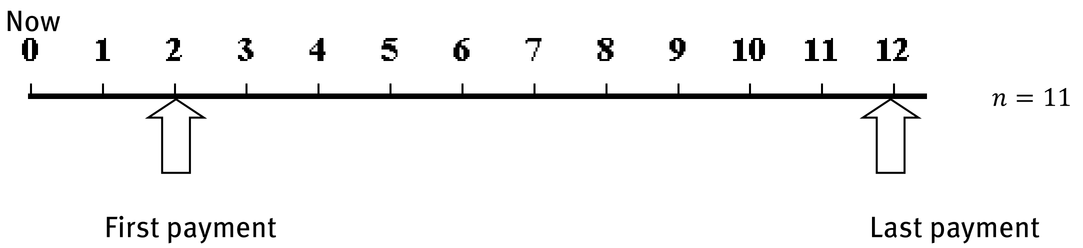
Example 2

First payment immediately. Last payment in one year's time.



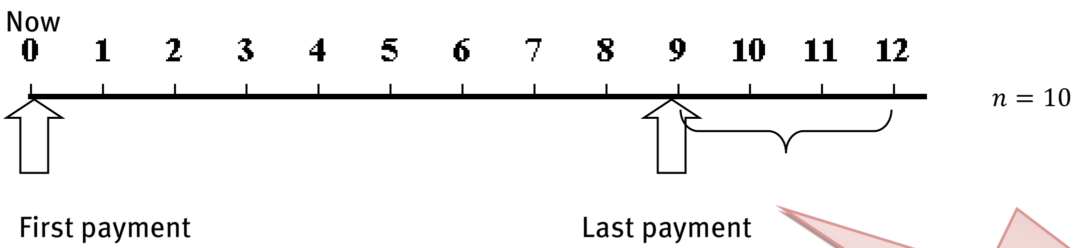
Example 3

Assume investment pays out in one year's time, but the first payment was made 2 months from now and the last payment in one year's time.



Example 4 **(Watch out!)**

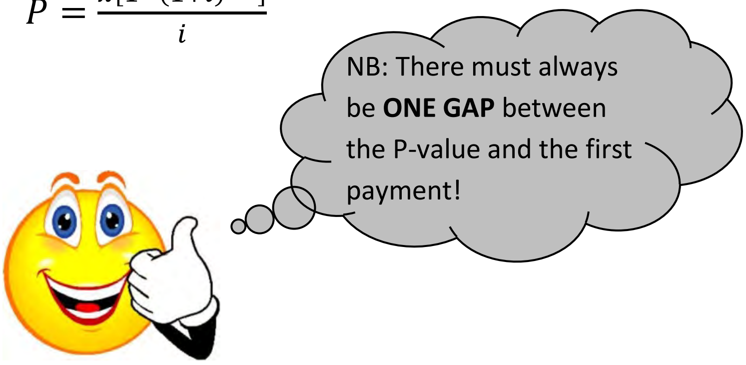
First payment immediately, but last payment in 9 months' time.



$$F = \frac{x[(1+i)^{10}-1]}{i}(1+i)^3$$

FUTURE VALUE ANNUITIES

$$P = \frac{x[1-(1+i)^{-n}]}{i}$$

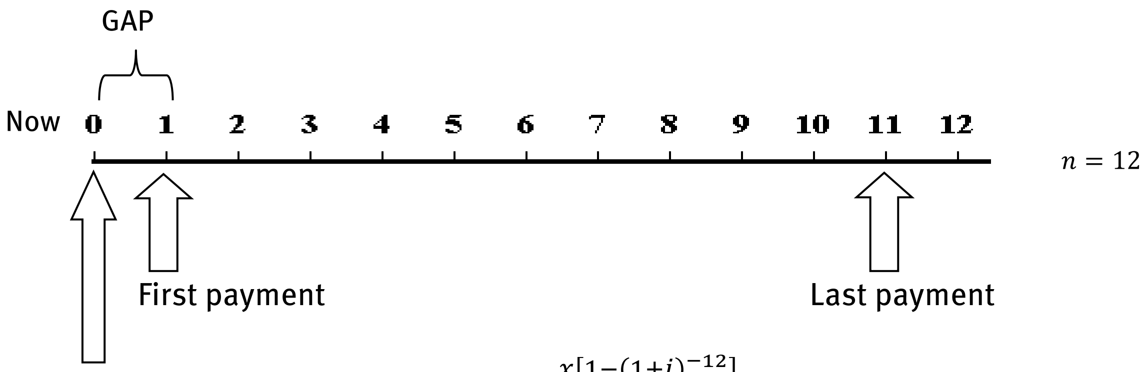


KEY WORDS:

- Regular payments (monthly/quarterly etc.)
- Loan (NOT HIREPURCHASE)
- Bond/home loan
- Repayment of debt
- How long will money be enough to provide regular income?

Example 1

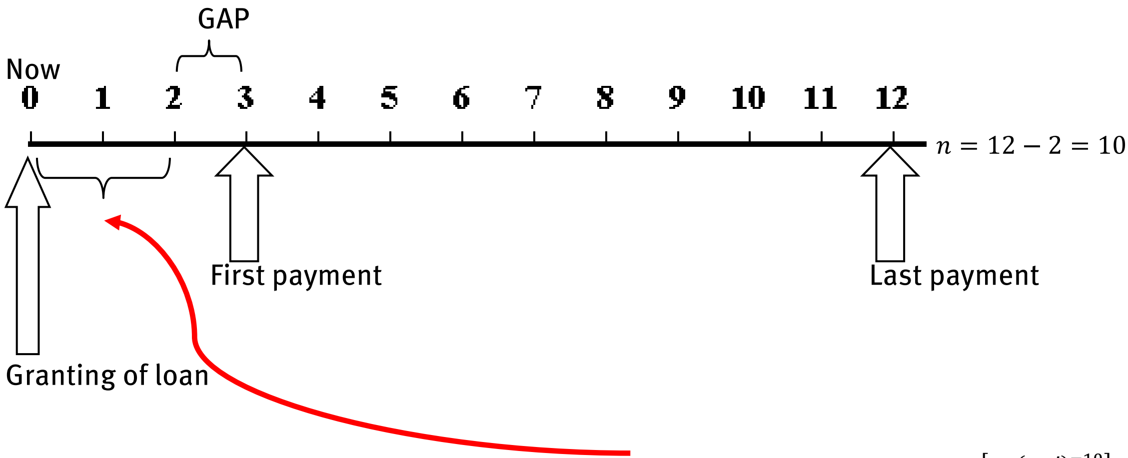
Payment starts one month after the granting of the loan. Last payment in one year's time.



$$P = \frac{x[1-(1+i)^{-12}]}{i}$$

Example 2

Payment starts in 3 months' time. Last payment in one year's time.



NB: Loan amount accumulates interest for **2 months**: $P(1+i)^2 = \frac{x[1-(1+i)^{-10}]}{i}$

OUTSTANDING BALANCE OF LOAN

Option 1
Use P-formula
n = **number of payments left**

Option 2
Use A- and F-formula
n = **number of payments**
already made

Example
A loan of is being repaid over 20 years in monthly payments of R6 000. The interest rate is 15% p.a. compounded monthly. What is the outstanding balance after 12½ years?

Option 1

Outstanding period = 7½ years = **90** months

$$Balance = \frac{6000\left[1-\left(1+\frac{0,15}{12}\right)^{-90}\right]}{\frac{0,15}{12}}$$

Option 2

Payments already made = 12½X12=**150 payments already paid**

Outstanding balance $= A - F$

$$\text{Balance} = P\left(1+\frac{0,15}{12}\right)^{150} - \frac{6000\left[\left(1+\frac{0,15}{12}\right)^{150}-1\right]}{\frac{0,15}{12}}$$ where P is the initial loan amount.

Mixed Exercise on Finance, growth and decay

1 Determine through calculation which of the following investments is the best, if R15 000 is invested for 5 years at:
   a 10,6% p.a. simple interest
   b 9,6% p.a., interest compounded quarterly.

2 An amount of money is now invested at 8,5% p.a compounded monthly to grow to R95 000 in 5 years.
   a Is 8,5% called the effective or nominal interest rate?
   b Calculate the amount that must be invested now.
   c Calculate the interest earned on this investment.

3 Shirley wants to buy a flat screen TV. The TV that she wants currently costs R8 000.
   a The TV will increase in cost according to the rate of inflation, which is 6% per annum. How much will the TV cost in two years' time?
   b For two years Shirley puts R2 000 into her savings account at the beginning of every six month period (starting immediately). Interest on her savings is paid at 7% per annum, compounded six-monthly. Will she have enough to pay for the TV in two years' time? Show all your calculations.

4 Calculate:
   a the effective interest rate to 2 dec. places if the nominal interest rate is 7,85% p.a., compounded monthly.
   b the nominal interest rate if interest on an investment is compounded quarterly, using an effective interest rate of 9,25% p.a.

5 Equipment with a value(new) of R350 000 depreciated to R179 200 after 3 years, based on the reducing balance method. Determine the annual rate of depreciation.

6 R20 000 is deposited into a new savings account at 9,75% p.a., compounded quarterly. After18 months, R10 000 more is deposited. After a further 3 months, the interest rate changes to 9,95% p.a., compounded monthly. Determine the balance in the account 3 years after the account was opened.

7 A company recently bought new equipment to the value of R900 000 which has to be replaced in 5 years' time. The value of the equipment depreciates at 15% per year according to the reduced-balance method. After 5 years the equipment can be sold second hand at the reduced value. The inflation rate on the equipment is 18% per year.

a The company wants to establish a sinking fund to replace the equipment in 5 years' time. Calculate what the value of the sinking fund should be to replace the equipment.

b Calculate the quarterly amount that the company has to pay into the sinking fund to be able to replace the equipment in 5 years' time. The company makes the first payment immediately and the last payment at the end of the 5 year period. The interest rate for the sinking fund is 8% per year compounded quarterly.

8 Goods to the value of R1 500 is bought on hire purchase and repaid in 24 monthly payments of R85. Calculate the annual interest rate that applied for the hire purchase agreement.

9 Peter makes a loan to buy a house. He pays back the loan over a period of 20 years in monthly payments of R6 500. Peter qualifies for an interest rate of 12% per years compounded monthly. He makes his first payment one month after the loan was granted.
a Calculate the amount Peter borrowed.
b Calculate the amount that Peter still owes on his house after he has been paying back the loan for 8 years.

10 Megan's father wants to make provision for her studies. He starts paying R1000 on a monthly base into an investment on her 12th birthday. He makes the last payment on her 18th birthday. She needs the money 5 months after her 18th birthday. The interest rate on the investment is 10% per annum compounded monthly. Calculate the amount Megan has available for her studies.

11 Stephan starts investing R300 into an investment monthly, starting one month from now. He earns interest of 9% per annum compounded monthly. For how long must he make these monthly investments so that the total value of his investment is R48 000? Give your answer as follows: …. years and …. Months

12 Carl purchases sound equipment to the value of R15 000 on hire purchase. The dealer expects him to put down a 10% deposit. The interest rate is 12% per annum and he has to repay the money monthly over 4 years. It is compulsory for him to insure the equipment through the dealer at a premium of R30 per month. Calculate the total amount Carl has to pay the dealer monthly.

13 Tony borrows money to the value of R400 000. He has to pay back the money in 16 quarterly payments, but only has to make his first payment one year from now. The interest rate is 8% per annum compounded quarterly. Calculate the quarterly payment Tony has to pay.

Chapter 5

Compound angles

Overview

Chapter 5 Page 102 Compound angles	**Unit 1 Page 108** Deriving a formula for $cos(\alpha - \beta)$	• How to deriving a formula for $cos(\alpha - \beta)$
	Unit 2 Page 112 Formulae for $\cos(\alpha + \beta)$ and $\sin(\alpha \pm \beta)$	• Formula for $\cos(\alpha + \beta)$ • Formula for $\sin(\alpha + \beta)$ • Formula for $\sin(\alpha - \beta)$
	Unit 3 Page 116 Double angles	• Formula for $\sin 2\alpha$ • Formula for $cos 2\alpha$
	Unit 4 Page 120 Identities	• Proving identities • Finding the value(s) for which the identity is not defined
	Unit 5 Page 124 Equations	• Equations with compound and double angles
	Unit 6 Page 128 Trigonometric graphs and compound angles	• Drawing and working with graphs of compound angles

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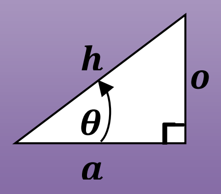
Chapter 5 Compound angles

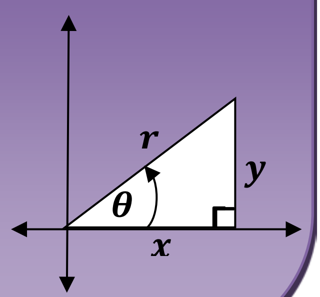
REVISION OF TRIGONOMETRY

BASIC TRIGONOMETRIC RATIOS

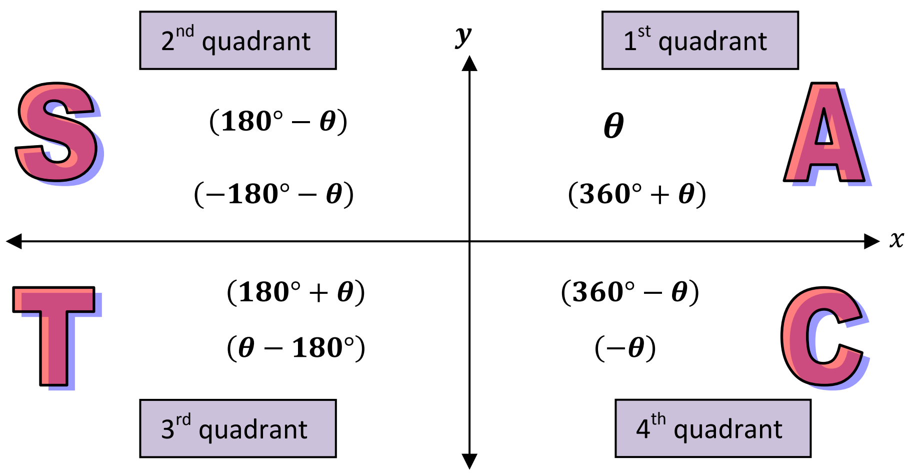
Ratio	Inverse
$sin\theta = \frac{o}{h}$	$cosec\theta = \frac{h}{o}$
$cos\theta = \frac{a}{h}$	$sec\theta = \frac{h}{a}$
$tan\theta = \frac{o}{a}$	$cot\theta = \frac{a}{o}$



Ratio	Inverse
$sin\theta = \frac{o}{h}$	$cosec\theta = \frac{h}{o}$
$cos\theta = \frac{a}{h}$	$sec\theta = \frac{h}{a}$
$tan\theta = \frac{o}{a}$	$cot\theta = \frac{a}{o}$



YOU HAVE TO KNOW IN WHICH QUADRANT AN ANGLES LIES AND WHICH RATIO (AND ITS INVERSE) IS POSITIVE THERE:



Chapter 5
Compound angles

REDUCTION FORMULAE

$sin(180° - \theta)$	$sin\theta$
$cos(180° - \theta)$	$-cos\theta$
$tan(180° - \theta)$	$-tan\theta$

$sin(180° + \theta)$	$-sin\theta$
$cos(180° + \theta)$	$-cos\theta$
$tan(180° + \theta)$	$tan\theta$

$sin(360° - \theta)$	$-sin\theta$
$cos(360° - \theta)$	$cos\theta$
$tan(360° - \theta)$	$-tan\theta$

CO-RATIOS/CO-FUNCTIONS

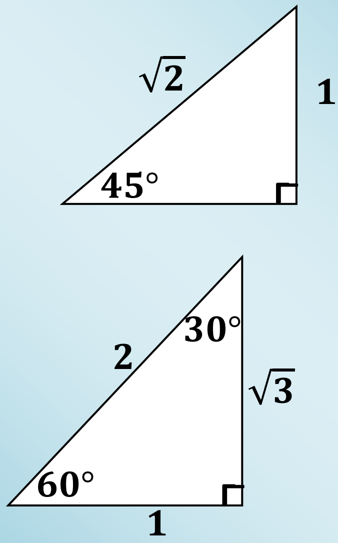
Ratio	Co-ratio
$sin(90° - \theta)$	$cos\theta$
$cos(90° - \theta)$	$sin\theta$
$tan(90° - \theta)$	$cot\theta$

$(90° - \theta)$
Is in 1st quadrant

Ratio	Co-ratio
$sin(90° + \theta)$	$cos\theta$
$cos(90° + \theta)$	$-sin\theta$
$tan(90° + \theta)$	$-cot\theta$

$(90° + \theta)$
Is in 2nd quadrant

KNOW YOUR SPECIAL TRIANGLES!



IDENTITIES

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \text{ and } \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

SQUARE IDENTITIES:

$$\sin^2\theta + \cos^2\theta = 1$$

From this follows that:

$$\therefore \cos^2\theta = 1 - \sin^2\theta$$
$$\therefore \sin^2\theta = 1 - \cos^2\theta$$

Note that the two identities above can both be FACTORISED as differences of two squares:

$$\cos^2\theta = 1 - \sin^2\theta = (1 - \sin\theta)(1 + \sin\theta)$$
$$\sin^2\theta = 1 - \cos^2\theta = (1 - \cos\theta)(1 + \cos\theta)$$

Compound angles

COMPOUND ANGLE-IDENTITIES

$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$
$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$
$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$
$$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$$

DOUBLE ANGLE-IDENTITIES

$$\sin 2\alpha = 2 \sin\alpha \cos \alpha$$

$$\begin{aligned}\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\ &= 1 - 2\sin^2\alpha \\ &= 2\cos^2\alpha - 1\end{aligned}$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2\alpha}$$

TIPS FOR PROVING IDENTITIES

- Work with LHS and RHS separately
- Write DOUBLE angles as SINGLE angles
- Watch out for SQUARE IDENTITIES
- Write everything in terms of sin and cos
- When working with fractions, put EVERYTHING over the LCD
- Be on the look out for opportunities to FACTORISE, e.g.
  - $2\sin\alpha\cos\alpha - \sin\alpha = \sin\alpha(2\cos\alpha - 1)$
  - $\cos^2\alpha - \sin^2\alpha = (\cos\alpha + \sin\alpha)(\cos\alpha - \sin\alpha)$
  - $2\sin^2\alpha + \sin\alpha - 1 = (2\sin\alpha - 1)(\sin\alpha + 1)$
- It is sometimes necessary to replace 1 with $\sin^2\alpha + \cos^2\alpha$

  E.g. $\sin 2\alpha + 1 = 2\sin\alpha\cos\alpha + \sin^2\alpha + \cos^2\alpha$
  $= (\sin\alpha + \cos\alpha)^2$

FINDING THE GENERAL SOLUTION OF A TRIGONOMETRIC EQUATION		
STEP		**EXAMPLES OF HOW TO APPLY STEP**
Get trig ratio (sin/cos/tan) alone on LHS **One value alone on RHS**	A B C	$2\sin 3x = 0{,}4$ $sin3x = 0{,}2$ $\frac{1}{3}cosx = -0{,}2$ $cosx = -0{,}6$ $2\tan(x-10°)+3=0$ $\tan(x-10°) = -\frac{3}{2}$
Now use RHS consisting of a: **SIGN (+ or -) and a VALUE** **Indicates Quadrant** **Get reference angle using:** $\boldsymbol{sin^{-1}(+value)}$ $\boldsymbol{cos^{-1}(+value)}$ **Or** $\boldsymbol{tan^{-1}(+value)}$	A B C	$sin3x = +0{,}2$ The $+$ indicates the 1st and 2nd quadrant, where sin is positive. Reference $\angle = sin^{-1}(0{,}2) = 11{,}54°$ $cosx = -0{,}6$ The – indicates the 2nd and 3rd quadrant, where cos is negative. Reference $\angle = cos^{-1}(0{,}6) = 53{,}13°$ $\tan(x-10°) = -\frac{3}{2}$ The $-$ indicates the 2nd and 4th quadrant, where tan is negative. Reference $\angle = tan^{-1}\left(\frac{3}{2}\right) = 56{,}31°$
The angle in the trig equations will be equated to the following in the respective quadrants: **1st** $=$ **Ref** $\angle$ **2nd** $=$ $\mathbf{180°}$ $-$**Ref** $\angle$ **3rd** $=$ $\mathbf{180°}$ $+$**Ref** $\angle$ **4th** $=$ $\mathbf{360°}$ $-$**Ref** $\angle$ $+k360°; k\in Z$ for sin/cos or $+k180°; k\in Z$ for tan **Then solve for** $\boldsymbol{x}$	A B C	$2\sin 3x = 0{,}4$ $sin3x = 0{,}2$ 1st: $3x = 11{,}54° + k360°\,;\ k\in Z$ $x = 3{,}85° + k120°$ OR 2nd: $3x = 180° - 11{,}54° + k360°$ $x = 56{,}15° + k120°$ $\frac{1}{3}cosx = -0{,}2$ $cosx = -0{,}6$ 2nd: $x = 180° - 53{,}13° + k360°;\ k\in Z$ $x = 126{,}87° + k360°$ OR 3rd: $x = 180° + 53{,}13° + k360°$ $x = 233{,}13° + k360°$ $2\tan(x-10°)+3=0$ $\tan(x-10°) = -\frac{3}{2}$ 2nd: $x - 10° = 180° - 56{,}31° + k180°;$ $k\in Z$ $x = 133{,}69° + k180°$

EQUATIONS INVOLVING TWO TRIGONOMETRIC FUNCTIONS	
EXAMPLES	COMMENTS
1 $\boldsymbol{sinx = cosx}$ $\boldsymbol{\frac{sinx}{cosx} = \frac{cosx}{cosx}}$ $\boldsymbol{tanx = 1}$ $\boldsymbol{x = 45° + k.180°\ ;k\epsilon Z}$	$\div$ by $cosx$ on both sides
2 $\boldsymbol{sinx = cos3x}$ $\boldsymbol{cos(90° - x) = cos3x}$ $\boldsymbol{90° - x = 3x + k.360°\ ;k\epsilon Z}$ $\boldsymbol{-4x = -90° + k.360°}$ $\boldsymbol{x = 22,5° - k.90°}$ **or** $\boldsymbol{90° - x = -3x + k.360°\ ;k\epsilon Z}$ $\boldsymbol{2x = -90° + k.360°}$ $\boldsymbol{x = -45° + k.180°}$	May NOT divide by $cosx$ both sides Trig function on both sides should be the same Angles on LHS and RHS should either be the same or be in two different quadrants where cos have the same sign (1st and 4th quadrant)
3 $\boldsymbol{\sin(x + 20°) = \cos(2x - 30°)}$ $\boldsymbol{cos[90° - (x + 20°)] = cos(2x - 30°)}$ $\boldsymbol{cos(70° - x) = cos(2x - 30°)}$ $\boldsymbol{70° - x = 2x - 30° + k.360°}$ $\boldsymbol{-3x = -100° + k.360°}$ $\boldsymbol{x = 33,33° - k.120°\ \ ;k \in Z}$ **or** $\boldsymbol{70° - x = -(2x - 30°) + k.360°}$ $\boldsymbol{x = -40° + k.360°}$	Alternative: sin on both sides $sin(x + 20°) = cos(2x - 30°)$ $sin(x + 20°) = \sin[90° - (2x - 30°)]$ $sin(x + 20°) = sin(120° - 2x)$ $x + 20° = 120° - 2x + k.360°$ $3x = 100° + k.360°$ $x = 33{,}33° - k.120°\ ;\ k \in Z$ or $x + 20° = 180° - (120° - 2x) + k.360°$ $-x = 40° + k.360°$ $x = -40° - k.360°$

EXAMPLES OF EQUATIONS INVOLVING DOUBLE ANGLES

$$\cos\theta . \cos 14^{o} + \sin\theta . \sin 14^{o} = 0,715$$
$$\cos(\theta - 14^{o}) = 0,715$$

Ref $\angle = 44,36^{o}$

1st quadrant:

$$\theta - 14^{o} = 44,36^{o} + k.360^{o}$$
$$\theta = 58,36^{o} + k.360^{o}; k \in Z$$

4th quadrant:

$$\theta - 14^{o} = -44,36^{o} + k.360^{o}$$
$$\theta = -30,36^{o} + k.360^{o}; k \in Z$$

$$\sin 2\theta + 2\sin\theta = 0$$
$$2\sin\theta\cos\theta + 2\sin\theta = 0$$
$$2\sin\theta(\cos\theta + 1) = 0$$

$\sin\theta = 0$ **or** $\cos\theta = -1$

$\theta = k.180^{o}; k \in Z$ **or** $\theta = 180^{o} + k.360^{o}$

$$2\sin^2\theta + \sin\theta = 3$$
$$2\sin^2\theta + \sin\theta - 3 = 0$$
$$\therefore (2\sin\theta + 3)(\sin\theta - 1) = 0$$

$2\sin\theta + 3 = 0$ **or** $\sin\theta + 1 = 0$

$\sin\theta = -\frac{3}{2}$ **or** $\sin\theta = -1$

No solution $\theta = 270^{o} + k.360^{o}; k \in Z$

PROBLEMS WITH COMPOUND-ANGLES TO BE DONE WITHOUT A CALCULATOR

- **Write given information in form where trig function is ALONE on LHS**
- **Select QUADRANT and draw TRIANGLE in correct quadrant (2 sides of triangle will be known)**
- **Use the Theorem of PYTHAGORAS to determine 3rd side**
- **Now work with the expression of which you need to find the value: write all compound or double angles in terms of SINGLE ANGLES**
- **Now SUBSTITUTE VALUES from diagram(s) and SIMPLIFY**

Example:

If $13sinx\alpha + 12 = 0$ and $\alpha \in [90°; 270°]$ and $\beta = \frac{5}{13}$; $\beta > 90°$, determine without the use of a calculator the value of:

a $sin(\alpha - \beta)$ **b** $cos(\alpha + \beta)$

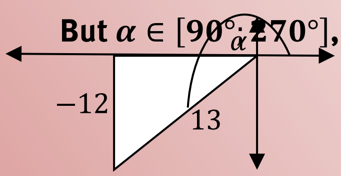
c $sin2\alpha$ **d** $cos2\beta$

Solution:

$sin\alpha = -\frac{12}{13}$

sin **negative in 3rd and 4th quad**

But $\alpha \in [90°; 270°]$, so 3rd quad

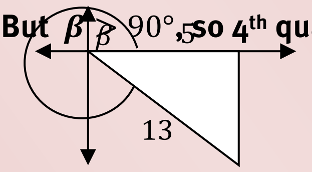


$x = -5$

$cos\beta = \frac{5}{13}$

cos **positive in 1st and 4th quad**

But $\beta > 90°$, so 4th quadrant



$y = -12$

a $sin(\alpha - \beta) = sin\alpha\ sin\beta - cos\alpha\ cos\beta = \left(\frac{-12}{13}\right)\left(\frac{5}{13}\right) - \left(\frac{-5}{13}\right)\left(\frac{-12}{13}\right) = \frac{-120}{169}$

b $cos(\alpha + \beta) = cos\alpha\ cos\beta - sin\alpha\ sin\beta = \left(\frac{-5}{13}\right)\left(\frac{5}{13}\right) - \left(\frac{-12}{13}\right)\left(\frac{-12}{13}\right) = -1$

c $sin2\alpha = 2sin\alpha\ cos\alpha = 2\left(\frac{-12}{13}\right)\left(\frac{-5}{13}\right) = \frac{120}{169}$

Mixed Exercise on Compound angles

1 Solve the following equations for . Give the general solution unless otherwise stated. Answers should be given correct to 2 decimal places where exact answers are not possible.

a $2cos2x + 1 = 0$

b $sin\, x = 3\cos x$ for $x \in [90°; 360°]$

c $sinx = \cos 3x$

d $6 - 10cosx = 3sin^2x$; $x \in [-360°; 360°]$

e $2 - sinx\ cosx - 3cos^2x = 0$

f $3sin^2x - 8sinx + 16sinx\ cosx - 6cosx + 3cos^2x = 0$

2 Prove the following identities, stating any values of x or θ for which the identity is not valid:

a $\cos x + \tan x \sin x = \dfrac{1}{\cos x}$

b $\dfrac{\sin\theta}{1-\cos\theta} - \dfrac{\cos\theta}{\sin\theta} = \dfrac{1}{\sin\theta}$

c $\dfrac{1-\cos^2 x}{\cos x} = \tan x \sin x$

d $\dfrac{\sin^3 x + \sin x \cos^2 x}{\cos x} = \tan x$

e $\dfrac{1+\tan x}{1-\tan x} = \dfrac{1+2\sin x\cos x}{\cos^2 x - \sin^2 x}$

f $\sin(45° + x) . \sin(45° - x) = \frac{1}{2}cos2x$

g $\dfrac{\sin 2\theta - \cos\theta}{\sin\theta - \cos 2\theta} = \dfrac{\cos\theta}{1+\sin\theta}$

h $\dfrac{\cos x - \cos 2x + 2}{3\sin x - \sin 2x} = \dfrac{1+\cos x}{\sin x}$

3 Simplify:

a $\dfrac{\sin(180^0 - x)\tan(-x)}{\tan(180^0 + x)\cos(x - 90^0)}$

b $\dfrac{\sin(180^0 + x)\tan(x - 360^0)}{\tan(360^0 - x)\cos 240^0 \tan 225^0}$ (without using a calculator)

4 Given that $sin17° = k$, express in terms of k:

a $\cos 73^o$

b $cos(-163^o)$

c $tan197^o$

d $cos326^o$

5 Given that $5cosx + 4 = 0$, calculate, without the use of a calculator, the value(s) of :

a $5sinx + 3tanx$

b $\tan 2x$

6 If $3sinx = -1$; $x \in [90°; 270°]$ and $tany = \frac{3}{4}$; $y \in [90°; 360°]$. Determine without the use of a calculator the value of:

a $\cos(x - y)$

b $cos2x - cos2y$

7 Simplify without the use of calculator:

a $cos^2 22{,}5° - sin^2 22{,}5°$

b $sin22{,}5°\ cos22{,}5°$

c $2sin15°cos15°$

Chapter 6

Solving problems in three dimensions

Overview

Chapter 6 Page 142 Solving problems in three dimensions	**Unit 1 Page 146**	
	Problems in three dimensions	• Trigonometry in real life
	Unit 2 Page 150	
	Compound angle formulae in three dimensions	• Using compound angle formulae in three dimensions

REMEMBER YOUR STUDY APPROACH SHOULD BE:

1. Work through all examples in this chapter of your Learner's Book.
2. Work through the notes in this chapter of this study guide.
3. Do the exercises at the end of the chapter in the Learner's Book.
4. Do the mixed exercises at the end of this chapter in the study guide.

The only way to STUDY Maths is to DO Maths!

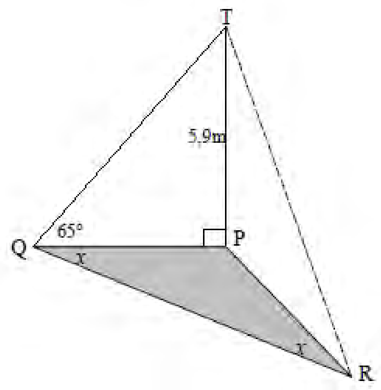
REVISION ON THE USE OF THE $sinus -$, $cosinus -$ and the $area -$FORMULAE			
INFORMATION GIVEN	UNKNOWN	FORMULA TO USE	FORM OF FORMULA
2 angles and 1 side (∠∠s)	s	sin-rule	$\frac{a}{sinA} = \frac{b}{\sin B}$ a is unknown
2 sides and a not- included ∠ (ss∠)	∠	sin-rule **Watch out** for ambiguous case! ∠ can be acute or obtuse	$\frac{sinA}{a} = \frac{sinB}{b}$ A is unknown
2 sides and an included ∠ (s∠s)	s	cos-rule	$a^2 = b^2 + c^2 - 2bccosA$ a is unknown
3 sides (sss)	∠	cos-rule	$cosA = \frac{b^2 + c^2 - a^2}{2bc}$ A is unknown
2 sides and an included ∠	Area	Area-rule	**Area** of $\Delta = \frac{1}{2}absinC$ **Area** is unknown
Area, side and ∠	s	Area-rule	$b = \frac{2 \times Area}{asinC}$ b is unknown

Solving problems in three dimensions

TIPS FOR SOLVING PROBLEMS IN THREE DIMENSIONS

- **Where there are 3 triangles, start with the Δ with the most information and work via the 2nd Δ to the 3rd Δ which contains the unknown to be calculated.**
- **Indicate all RIGHT angles – remember they don't always look like $90°$ angles**
- **Shade the horizontal plane in the diagram (e.g. floor, ground)**
- **Be on the lookout for reductions like $\cos(90° - \alpha) = \sin\alpha$ and $\sin(180° - \alpha) = \sin\alpha$ to simplify expressions**
- **Use compound and double angle formulae to convert to single angles**
- **When writing out the solution – always indicate in which Δ you are working**

EXAMPLE



P, Q and R are in the same horizontal plane. TP is a vertical tower 5,9 m high. The angle of elevation of T from Q is 65°. $P\hat{Q}R = P\hat{R}Q$.

a Calculate the length of PQ to the nearest meter.

b Hence show that $RQ = 5{,}5\cos x$.

c If it is further given that $x = 42°$, calculate the area of ΔPQR.

Solution:

a $\frac{5{,}9}{PQ} = \tan 65°$

$\therefore PQ = \frac{5{,}9}{\tan 65°} = 2{,}75\ m$

b $Q\hat{P}R = 180° - 2x$

$\frac{RQ}{\sin P} = \frac{PQ}{\sin R}$

$\frac{RQ}{\sin(180° - 2x)} = \frac{2{,}75}{\sin x}$

$\frac{RQ}{\sin 2x} = \frac{2{,}75}{\sin x}$

$\frac{RQ}{2\sin x\cos x} = \frac{2{,}75}{\sin x}$

$\therefore RQ = 2 \times 2{,}75\cos x$

$RQ = 5{,}5\cos x$

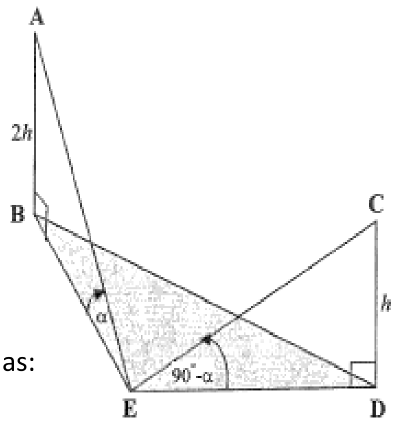
c Area of $\Delta PQR = \frac{1}{2} \times PQ \times QR \times \sin Q$

$= \frac{1}{2} \times 2{,}75 \times (5{,}5\cos 42°) \times \sin 42°$

$= 3{,}76$ square units

Mixed Exercise on Problems in Three Dimensions

1 In the diagram alongside B, D and E are in the same horizontal plane. $B\hat{E}D = 120°$
AB and CD are two vertical towers.
AB = 2CD = $2h$ meter
The angle of elevation from E to A is α.
The angle of elevation from E to C is $(90° - \alpha)$.



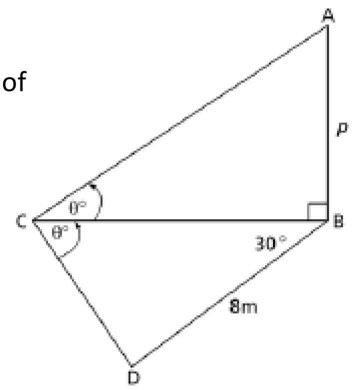
a Determine the length BE in terms of h and α.

b Show that the distance between the two towers can be given as:

$$BD = \frac{h\sqrt{tan^4\alpha + 2tan^2\alpha + 4}}{tan\alpha}$$

c Hence determine the height of the tower CD, rounded to the nearest meter, if $\alpha = 42°$ and BD=400 m.

2 B, C and D are three points in the same horizontal plane and AB is a vertical pole of length p metres. The angle of elevation of A from C is θ and $B\hat{C}D = \theta$. Also, $C\hat{B}D = 30°$ and $BD = 8\ m$.

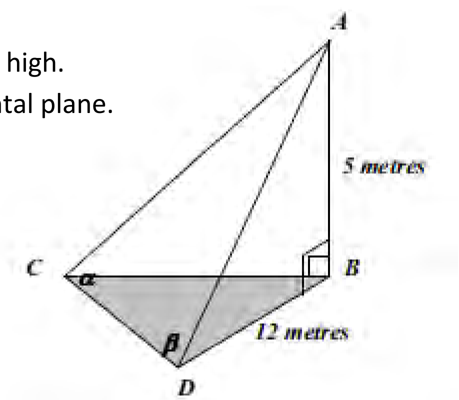


a Express $C\hat{D}B$ in terms of θ.

b Hence show that $p = \frac{8\sin(30°+\theta)}{cos\theta}$.

3 In the diagram alongside, AB is a vertical flagpole 5 metres high. AC an AD are two stays. B, C and D are in the same horizontal plane. $BD = 12\ m$, $A\hat{C}D = \alpha$ and $A\hat{D}C = \beta$.

Show that $CD = \frac{13\sin(\alpha+\beta)}{sin\alpha}$



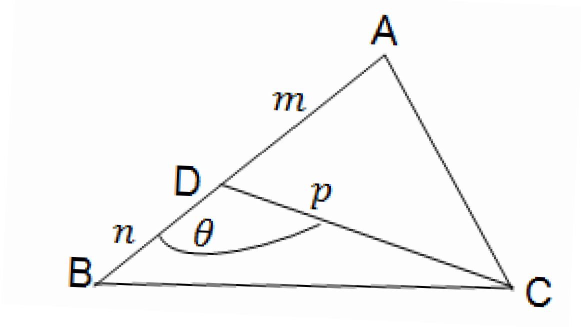
4 In ΔABC AD = ; DB = n ; CD = p and $B\widehat{D}C=\theta$.

a Complete in terms of m , p and θ: Area $\Delta ADC = \cdots$

b Show that the area of $\Delta ABC = \frac{1}{2}p(m+n)\sin\theta$.

c If the area of $\Delta ABC = 12{,}6\ cm^2$; $AB = 5{,}9\ cm$

and $DC = 8{,}1\ cm$, calculate the value(s) of θ.



5 In the diagram, $\widehat{D} = 90^o$, $B\hat{C}D = \theta$

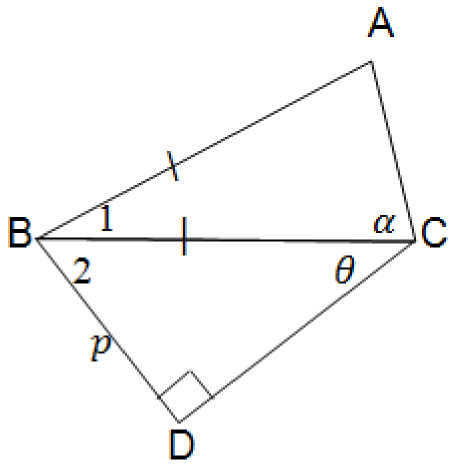
$A\hat{C}B = \alpha$; AB = BC and $BD = p$ units.

a Express BC in terms of p and θ.

b Determine, without stating reasons,

the size of $\widehat{B}_1$ in terms of α.

c Hence , prove that AC = $\frac{p.\sin 2\alpha}{\sin\theta.\sin\alpha}$



6 In the diagram PQ is a vertical building.

Q, R and S are points in the same horizontal plane.

The angle of elevation of P, the top of the building,

measured from R, is α.

$R\hat{Q}S = 30^o$

$QSR = 150^o - \alpha$

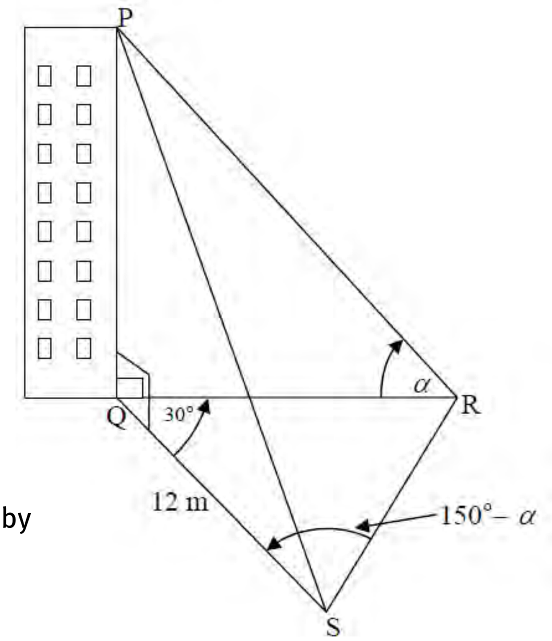
$QS = 12\ m$

a Show that $QR = \frac{6(cos\alpha+\sqrt{3}sin\alpha)}{sin\alpha}$

b Hence show that the height PQ of the building is given by

$$PQ = 6 + 6\sqrt{3}tan\alpha$$

c Hence calculate the value of α if PQ = 23 m.



Chapter 7

Polynomials

Overview

Chapter 7 Page 156 Polynomials	Unit 1 Page 158	
	The Remainder Theorem	• The Remainder theorem
	Unit 2 Page 160	
	The Factor Theorem	• The Factor Theorem

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THE REMAINDER THEOREM

$$f(x) = (ax+b).q(x) + r(x)$$

divisor quotient remainder

The remainder theorem can be used to calculate the remainder when a polynomial $f(x)$is divided by $(ax+b)$

$$\therefore f\left(-\frac{b}{a}\right) = r(x)$$

Choosing the **correct value to substitute** is very important:

If you divide f by $g(x) =$	Value to substitute into $f(x)$
$(\boldsymbol{x-2})$	$f(2) =?$
$(\boldsymbol{2x-1})$	$f\left(\frac{1}{2}\right) =?$
$(\boldsymbol{x+3})$	$f(-3) =?$
$(\boldsymbol{3x+2})$	$f\left(-\frac{2}{3}\right) =?$

THE FACTOR THEOREM

If $f\left(-\frac{b}{a}\right) = 0$ then:

- $(ax+b)$ is a **FACTOR** of $f(x)$ and
- $f(x)$ is **DIVISIBLE** by $(ax+b)$

When trying out x –values that give 0, try them in the following order: 1: -1: 2: -2: 3: -3 etc.

DIFFERENT METHODS TO FACTORISE A CUBIC POLYNOMIAL (3RD DEGREE)	
METHOD AND DESCRIPTION OF STEPS	EXAMPLES
SUM AND DIFFERENCE OF CUBES	A) $f(x) = x^3 + 27$ $= (x+3)(x^2 - 3x + 9)$ Cannot factorise further B) $f(x) = 8x^3 - 1$ $= (2x-1)(4x^2 + 2x + 1)$ Cannot factorise further
FACTORISE BY GROUPING • **Group terms in two pairs** • **Take out common factor from each pair** • **Two sets of brackets now become common factor** • **Factorise bracket further if possible**	$f(x) = \mathbf{x^3 + 3x^2 - 4x - 12}$ $= x^2\mathbf{(x+3)} - 4\mathbf{(x+3)}$ $= \mathbf{(x+3)(x^2 - 4)}$ $= (x+3)\mathbf{(x+2)(x-2)}$
FACTORISE BY INSPECTION • **Find one linear factor using factor theorem** • **Find other factor (quadratic expression) by inspection**	$f(x) = 2x^3 \mathbf{- 2x^2} - 10x - 6$ $f(-1) = 2(-1)^3 - 2(-1)^2 - 10(-1) - 6 = 0$ $\therefore (x+1)$ is a factor $f(x) = (x+1)(\boldsymbol{a}x^2 + \boldsymbol{b}x + \boldsymbol{c})$ Now find these coefficients Start with $\boldsymbol{a}$ and $\boldsymbol{c}$: $1 \times a = 2 \quad \therefore a = 2$ $1 \times c = -6 \therefore c = 6$ You now need to find $\boldsymbol{b}$: Multiply the two brackets; the two x^2-terms need to give you $\mathbf{-2x^2}$: $f(x) = (x+1)(\mathbf{2}x^2 + \boldsymbol{b}x + \mathbf{6})$ $bx^2 + 2x^2 = \mathbf{-2x^2} \qquad \therefore b = -4$ $\therefore f(x) = (x+1)(2x^2 - 4x + 6)$ $= (x+1)(2x+2)(x-3)$
SYNTHETIC OR LONG DIVISION • **Find one linear factor using factor theorem** • **Find other factor (quadratic expression) by long division or synthetic division** **(SEE NEXT PAGE)**	$f(x) = 2x^3 \mathbf{- 2x^2} - 10x - 6$ $f(-1) = 2(-1)^3 - 2(-1)^2 - 10(-1) - 6 = 0$ $\therefore (x+1)$ is a factor $f(x) = (x+1)(\boldsymbol{a}x^2 + \boldsymbol{b}x + \boldsymbol{c})$ Find a, b, c using synthetic division

SYNTHETIC DIVISION

$f(-1) = 0$, so $(x+1)$is a factor

$f(-1) = 0$ $\qquad$ $f(x) = 2x^3 - 2x^2 - 10x - 6$

-1	$\mathbf{2}$	$\mathbf{-2}$	$\mathbf{-10}$	$\mathbf{-6}$
		$\mathbf{-2}$	$\mathbf{4}$	$\mathbf{6}$
	$\mathbf{2}$	$\mathbf{-4}$	$\mathbf{-6}$	$\mathbf{0}$

This method is called SYNTHETIC division, because we don't really divide.
We actually multiply and add.

Note the following:

- The x-value of -1 that gave us the factor $(x+1)$ is written on the LHS
- The coefficients of the cubic polynomial are written in the top row
- The first coefficient, 2, is carried down to the last row
- Now starting from the left:
  MULTIPLY along the dotted arrow
  and write the ANSWER in the block one row up and one column right
- Now ADD DOWN in the column (the two values underneath each other)

  repeat steps
- You MUST get 0 in the last block
- The 3 values in the bottom row are the coefficients of the quadratic factor.

So, $f(x) = (x+1)(2x^2 - 4x - 6)$

You can now complete the factorising:

$f(x) = (x+1)(2x+2)(x-3)$

Mixed Exercise on Polynomials

1 Factorise the following expressions completely:

a $27x^3 - 8$

b $5x^3 + 40$

c $x^3 + 3x^2 + 2x + 6$

d $4x^3 - x^2 - 16x + 4$

e $4x^3 - 2x^2 + 10x - 5$

f $x^3 + 2x^2 + 2x + 1$

g $x^3 - x^2 - 22x + 40$

h $x^3 + 2x^2 - 5x - 6$

i $3x^3 - 7x^2 + 4$

j $x^3 - 19x + 30$

k $x^3 - x^2 - x - 2$

2 Solve for x:

a $x^3 + 2x^2 - 4x = 0$

b $x^3 - 3x^2 - x + 6 = 0$

c $2x^3 - 12x^2 - x + 6 = 0$

d $2x^3 - x^2 - 8x + 4 = 0$

e $x^3 + x^2 - 2 = 0$

f $x^3 = 16 + 12x$

g $x^3 + 3x^2 = 20x + 60$

3 Show that $x - 3$ is a factor of $f(x) = x^3 - x^2 - 5x - 3$ and hence solve $f(x) = 0$.

4 Show that $2x - 1$ is a factor of $g(x) = 4x^3 - 8x^2 - x + 2$ and hence solve $g(x) = 0$.

Chapter 8

Differential calculus

Overview

Chapter 8 Page 166 Differential calculus	**Unit 1 Page 170** Limits	• Investigating limits
	Unit 2 Page 172 The gradient of a graph at a point	• Function notation and the average gradient • The gradient of a graph at a point
	Unit 3 Page 176 The derivative of a function	• First principles • Rules for differentiation • The derivative at a point
	Unit 4 Page 182 The equation of a tangent to a graph	• Calculating the equation of a tangent to a graph
	Unit 5 Page 184 The graph of the cubic function	• Plotting a cubic function
	Unit 6 Page 188 The second derivative (concavity)	• Change in concavity
	Unit 7 Page 192 Application of differential calculus	• Modelling real-life problems

REMEMBER YOUR STUDY APPROACH SHOULD BE:

1 Work through all examples in this chapter of your Learner's Book.

2 Work through the notes in this chapter of the study guide.

3 Do the exercises at the end of the chapter in the Learner's Book.

The only way to STUDY Maths is to DO Maths!

THE CONCEPT OF A LIMIT

Notation: $\lim_{x\to 4} f(x)$

We say: "The limit of f as x approaches 4"

What does it mean?

The limit is the y-value (remember $y = f(x)$) which the function approaches as the x-value approaches (gets closer to) a certain value from the left or the right .

Examples: a Let $f(x) = 2x^2 + 4$

$$\lim_{x\to 1} f(x) = \lim_{x\to 4} 2x^2 + 4$$

$$= 2(1)^2 + 4 = 6$$

Before calculating the limit, it is sometimes necessary to FACTORISE and SIMPLIFY first:

An examples of this is:

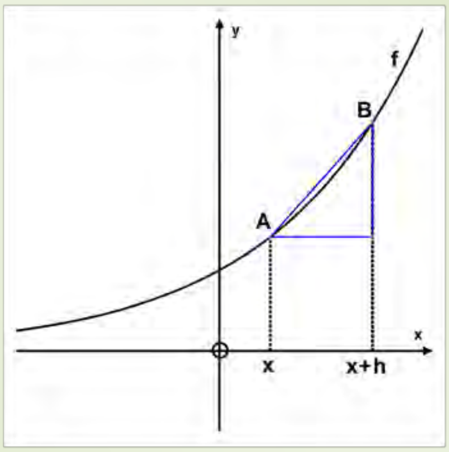
$\lim_{x\to 3} \frac{x^2-9}{x-3}$	Substituting $x = 3$ now will cause division by 0
$= \lim_{x\to 3} \frac{\cancel{(x-3)}(x+3)}{\cancel{x-3}}$	First factorise the numerator and cancel out
$= \lim\limits_{x\to 3}(x+3)$	
$= (3+3)$	Note that "lim" falls away in the step where you substitute
$= 6$	

AVERAGE GRADIENT BETWEEN 2 POINTS

From previous grades you know you can calculate the gradient between two points $(x_1; y_1)$ and $(x_2; y_2)$ using the formula: $m = \frac{y_2 - y_1}{x_2 - x_1}$

In the diagram below the points A$(x; f(x))$ and B$(x + h; f(x + h))$ are indicated.

The AVERAGE GRADIENT between A and B is given by: $m_{AB} = \frac{f(x+h) - f(x)}{h}$



THE GRADIENT OF A GRAPH AT A POINT

By letting h approach 0, the distance between point A and B will become smaller and smaller. A and B will almost "become one point".

The average gradient then becomes the

GRADIENT OF THE GRAPH AT A POINT $= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

NOTATION: This is denoted by $\boldsymbol{f'(x)}$

Differential calculus

The formula $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ can be used to find any of the following from **FIRST PRINCIPLES**:

Look out for the words:

FIRST PRINCIPLES

- The **derivative** of f at any point
- The **gradient** of the **tangent** to graph f at any point
- The **gradient** of the **function** f at any point
- The **rate of change** of f at any point

$f'(x)$ **can also be determined using DIFFERENTIATION RULES**		
Function f	Derivative $f'(x)$	Examples
$\boldsymbol{f(x) = k}$ where $\boldsymbol{k}$ is a constant	$f'(x) = 0$	• $f(x) = -5$ $f'(x) = 0$ • $y = 4$ $\frac{dy}{dx} = 0$
$\boldsymbol{f(x) = x^n; x \epsilon R}$	$f'(x) = nx^{n-1}$	• $D_x[x^6] = 6x^5$ • $f(x) = \frac{1}{x^3} = x^{-3}$ $f'(x) = -3x^{-4} = \frac{-3}{x^4}$
$\boldsymbol{f(x) = kx^m; m \in R}$ where $\boldsymbol{k}$ is a constant	$f'(x) = k \times mx^{m-1}$	• $f(x) = 2x^4$ $f'(x) = 2 \times 4x^{4-1}$ $= 8x^3$ • $D_x\left[\frac{1}{2}x^{\frac{5}{2}}\right] = \frac{1}{2} \times \frac{5}{2} x^{\frac{5}{2}-1}$ $= \frac{5}{4}x^{\frac{3}{2}}$
When functions are added/subtracted, apply the rule to each function separately: $\boldsymbol{D_x[f(x) \pm g(x)] = D_x[f(x)] \pm D_x[g(x)]}$		• $D_x[5x^2 - 4x + 6]$ $= 5 \times 2x - 4 + 0$ $= 10x - 4$

BEFORE YOU APPLY THE DIFFERENTIATION RULES, MAKE SURE THERE ARE:

- No brackets :

  a $\quad f(x) = (x+1)(2x-1) = 2x^2 + x - 1$

  $\qquad\qquad f'(x) = 4x + 1$

- No x under a fraction line:

  b $\quad f(x) = \frac{3x^2-2}{x} = \frac{3x^2}{x} - \frac{2}{x} = 3x - 2x^{-1}$

  $\quad f'(x) = 3 - 2(-1)x^{-2} = 3 + \frac{2}{x^2}$

  c $\quad f(x) = \frac{x^2-x-6}{x+2} = \frac{(x-3)(x+2)}{x+2} = x - 3$

  $\quad f'(x) = 1$

- No x under a root sign:

  d $\quad f(x) = 3\sqrt{x} - 4x = 3x^{\frac{1}{2}} - 4x$

  $\quad f'(x) = 3 \times \frac{1}{2} x^{\frac{1}{2}-1} - 4 = \frac{3}{2} x^{-\frac{1}{2}} - 4$

NB: NOTE THE DIFFERENCE BETWEEN THE FOLLOWING:

$f(4)$ is the y**-VALUE** of the function at $x = 4$

$f'(4)$ is the **GRADIENT** of the function at $x = 4$

As well as the gradient of the **TANGENT** at $x = 4$

THE CUBIC GRAPH $y = ax^3 + bx^2 + cx + d$

a indicates THE SHAPE

$a > 0\ (+)$

or

$a < 0(-)$

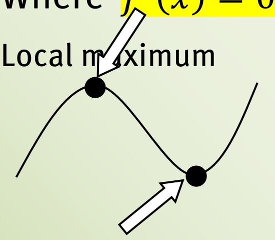
or

STATIONARY POINTS

TURNING POINTS

Where $f'(x) = 0$

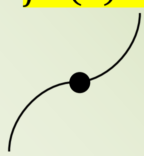
Local maximum



Local minimum

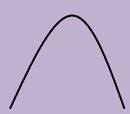
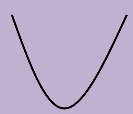
POINTS OF INFLECTION

Where $f''(x) = 0$



How do I determine whether it is:

A LOCAL MAXIMUM or A **LOCAL MINIMUM**?

What does f' and f'' tell me?			
	Negative (< 0)	$= 0$	Positive (> 0)
f'	f decreases	f turns	f increases
f''	Local maximum Concave down 	Point of inflection	Local minimum Concave up 

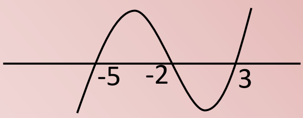
x –INTERCEPTS/ROOTS AND SHAPE

- For x –intercepts: solve $f(x) = 0$
- A cubic graph can have either one(only) or two or three x –intercepts

EXAMPLES:

a $f(x) = x^3 + 4x^2 - 11x - 30$

$f(3) = (3)^3 + 4(3) - 11(3) - 30 = 0$



$\therefore (x-3)$ is a factor

$f(x) = (x-3)(x^2 + 7x + 10)$

$= (x-3)(x+2)(x+5)$

The roots are -5; -2 and 3.

b $f(x) = x^3 - 3x + 2$

$f(1) = (1)^3 - 3(1) + 2 = 0$

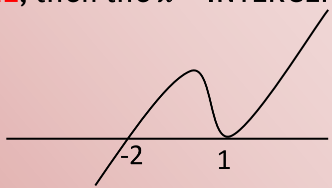
$\therefore (x-1)$ is a factor

$f(x) = (x-1)(x^2 + x - 2)$

$f(x) = (x-1)(x-1)(x+2) = (x-1)^2(x+2)$

If **TWO FACTORS** are the **SAME**, then the x –INTERCEPT is also a TURNING POINT.

The graph **BOUNCES** at $x = 1$



EQUATION OF TANGENT TO GRAPH AT A SPECIFIC POINT

- Substitute x –value **into $f(x)$** to find coordinates of POINT of TANGENCY
- Determine $f'(x)$ using differentiation rules
- Substitute x –value into $f'(x)$ to find GRADIENT of TANGENT, m
- Substitute gradient into $y = mx + c$
- Substitute point of tangency in $y = mx + c$ to find the value of c.

EXAMPLE

Determine equation of tangent to $f(x) = 2x^3 - 5x^2 - 4x + 3$ at $x = 1$

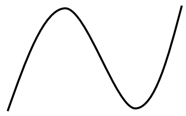
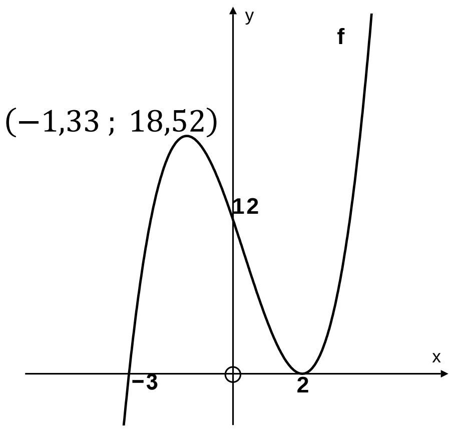
Substitute $x = 1$: $f(1) = 2(1)^3 - 5(1)^2 - 4(1) + 3 = -4$
$\therefore$ Point of tangency is $(1; -4)$

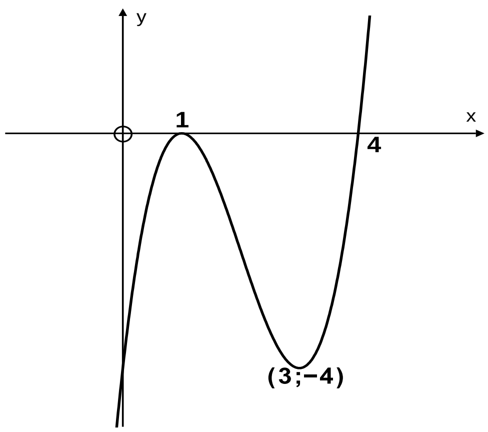
$f'(x) = 6x^2 - 10x - 4$
$f'(1) = 6(1)^2 - 10(1) - 4 = -8$
$\therefore$ Gradient of tangent at $x = 1$ is -8; so $y = -8x + c$

Substitute $(1; -4)$ into $y = -8x + c$: $\quad -4 = -8(1) + c$
$\therefore c = 4$

Equation of tangent is $y = -8x + 4$

SKETCHING THE CUBIC GRAPH
EXAMPLE: $f(x) = x^3 - x^2 - 8x + 12$

DESCRIPTION OF STEP	STEP APPLIED TO THIS EXAMPLE
Determine shape (using a)	$a = 1$ (positive) 
Determine y –intercept **Make $x = 0$**	$y = (0)^3 - (0)^2 - 8(0) + 12 = 12$
Determine x –intercepts **Solve $f(x) = 0$**	$f(2) = 0$ $\therefore (x - 2)$ is a factor $f(x) = (x - 2)(x^2 + x - 6)$ $f(x) = (x - 2)(x - 2)(x + 3)$ Roots are -3 and 2. $x = 2$ is also a turning point where graph **bounces**
Determine turning points and their y –values **Solve $f'(x) = 0$** **Substitute x –values into $f(x)$**	$f'(x) = 3x^2 - 2x - 8 = 0$ $(3x + 4)(x - 2) = 0$ $x = -\frac{4}{3}$ or $x = 2$ We already know from the previous step that $(2; 0)$ is one turning point (local minimum). Let us now find the other TP's y –coordinates $f(x) = \left(\frac{-4}{3}\right)^3 - \left(\frac{-4}{3}\right)^2 - 8\left(\frac{-4}{3}\right) + 12 = 18{,}52$ Local maximum at $(-1{,}33\ ;\ 18{,}52)$
Make a neat drawing	

FINDING THE EQUATION OF A CUBIC GRAPH IN THE FORM $y = ax^3 + bx^2 + cx + d$	
INFORMATION GIVEN (CAN BE SHOW ON A GRAPH OR NOT)	STEPS
y –intercept: $y = -8$ **and x –intercepts: $x = -2; -1$ and 4**	From the y –intercept we already know that $d = -8$. But we are going to use the three roots: $y = a(x+2)(x+1)(x-4)$ Substitute the point $(0; -8)$: $-8 = a(0+2)(0+1)(0-4)$ $-8 = -8a$ $a = 1$ $\therefore y = 1(x+2)(x+1)(x-4)$ Removing the brackets gives: $y = x^3 - x^2 - 10x - 8$
	We were given two roots (of which one is also a turning point) and the other turning point. NB: The graph BOUNCES at $x = 1$. This factor will therefore have to be squared. $y = a(x-1)^2(x-4)$ Substitute the other turning point $(3; -4)$: $-4 = a(3-1)^2(3-4)$ $-4 = -4a$ $a = 1$ $\therefore y = a(x-1)^2(x-4)$ Removing the brackets gives: $y = x^3 - 6x^2 + 9x - 4$

SPECIAL APPLICATIONS OF DERIVATIVES

RATES OF CHANGE

- Distance/Displacement $s(t)$
- Speed/Velocity $s'(t)$
- Acceleration $s''(t)$

EXAMPLE

The displacement of a moving object is described by the equation $s(t) = 10t - t^2$ where s, represents displacement in metres and t, time in seconds.

a Determine the displacement after 2 seconds.

b What time will it take for the object to reach a maximum displacement?

c Determine the velocity of the object after 3 seconds.

d Determine the acceleration of the object. Is it going faster or slower?

SOLUTIONS

a $s(2) = 10(2) - (2)^2 = 16\,m$

b $s'(t) = 10 - 2t = 0$

$\therefore 10 - 2t = 0$

$\therefore t = 5\,s$

c $s'(3) = 10 - 2(3) = 4\,m.s^{-1}$

d $s''(t) = -2\,m.s^{-2}$

The object is going slower because the acceleration is negative.

USING FIRST DERIVATIVE TO DETERMINE MINIMUM OR MAXIMUM

- For area, $A(x)$, to be a min/max, solve $A'(x) = 0$
- For volume, $V(x)$, to be a min/max, solve $V'4(x) = 0$
- For cost, $C(x)$, to be a minimum, solve $C'(x) = 0$
- For profit, $P(x)$, to be a maximum, solve $P'(x) = 0$

EXAMPLE

The volume of water in a water reservoir is given by: $V(t) = 60 + 8t - 3t^2$ where $V(t)$ is the volume in thousands of litres and t is the number of days water is pumped into the reservoir.

a Determine the rate of change of the volume after 3 days.

b When will the volume of water in the reservoir be a maximum?

c What will the maximum level of water in the reservoir be?

SOLUTIONS

a $V'(t) = 8 - 6t$

$\therefore V'(3) = 8 - 6(3) = -10$ thousand liters/day

b $V'(t) = 8 - 6t = 0$

$\therefore 8 - 6t = 0$

$t = \frac{4}{3} = 1{,}3$ days

c $V\left(\frac{4}{3}\right) = 60 + 8\left(\frac{4}{3}\right) - 3\left(\frac{4}{3}\right)^2 = 58{,}67$ thousand litres

Mixed Exercise on Differential Calculus

1 Determine f' from first principles if

a $f(x) = 1 - x^2$

b $f(x) = -3x^2$

2 Determine:

a $\frac{dy}{dx}$ if $y = \sqrt{x} - \frac{1}{2x^2}$

b $D_x\left[\frac{2x^2-x-15}{x-3}\right]$

3 Determine the equation of the tangent to the curve $f(x) = -2x^3 + 3x^2 + 32x + 15$ at the point $x = -2$.

4 Sketch the graph with the following properties showing all the key points on the graph:

$f'(x) < 0$ when $1 < x < 5$

$f'(x) > 0$ when $x < 1$ and $x > 5$

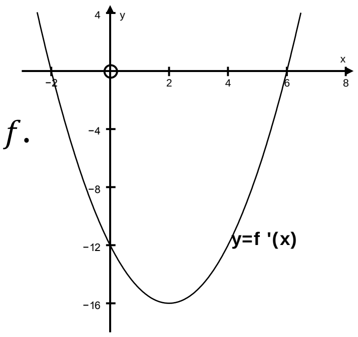
$f'(5) = 0$ and $f'(1) = 0$

$f(0) = -6$ and $f(3) = 0$

$f''(3) = 0$

5 $(2; 9)$ is a turning point of the graph $f(x) = ax^3 + 5x^2 + 4x + b$. Determine the values of a and b in the equation of f.

6 The diagram below represents the graph of $y = f'(x)$, the derivative of f.



a Write down the x-values of the turning point of f.

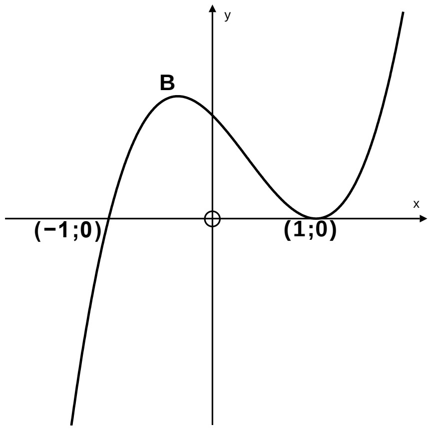
b Write down the x-value of the point of inflection of f.

c For which values of x will $f(x)$ decrease?

7 The graph of $f(x) = x^3 + ax^2 + bx + c$ is drawn. The curve has turning points at B and $(1; 0)$. The points $(-1; 0)$ and $(1; 0)$ are x-intercepts.

a Show that $a = -1;\ b = -1$ and $c = 1$.

b Determine the coordinates of B.



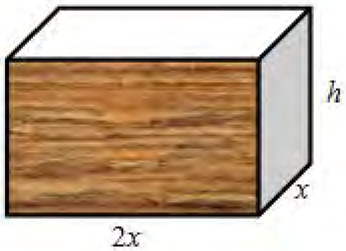
8 The distance covered in metres by an object is given as $s(t) = t^3 - 2t^2 + 3t + 5$. Determine:

a an expression for the speed of the object at any time t.

b the time at which the speed of the object is at a minimum.

c the time at which the acceleration of the object will be 8 $m.s^{-2}$.

9 The sketch below shows a rectangular box with base ABCD.
AB = $2x$ metres and BC = x metres. The volume of the box is 24 cubic metres.
Material to cover the top (PQRS) of the box costs R25 per square metre.
Material to cover the base ABCD and the four sides costs R20 per square metre.



a Show that the height(h) of the box is given by $h = 12x^{-2}$.

b Show that the total cost (C) in rand is given by: $C(x) = 90x^2 + 1440x^{-1}$.

c Determine the value of x for which the cost will be a minimum.

Chapter 9

Analytical geometry

Overview

Chapter 9 Page 202 Analytical geometry	**Unit 1 Page 204**	
	Equation of a circle with centre at the origin	• Finding the equation of a circle • Symmetrical points on a circle
	Unit 2 Page 208	
	Equation of a circle centred off the origin	• Finding the equation of a circle with any given centre • General form
	Unit 3 Page 214	
	The equation of the tangent to the circle	• Lines on circles • Equation of a tangent

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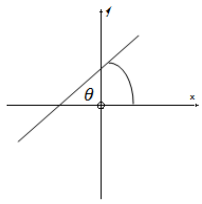
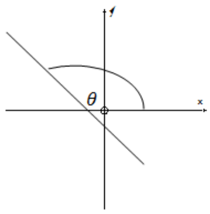
2. Work through the notes in this chapter of the study guide.
3. Do the exercises at the end of the chapter in the text book.
4. Do the mixed exercises at the end of this chapter in the study guide.

The only way to STUDY Maths is to DO Maths!

Chapter 9

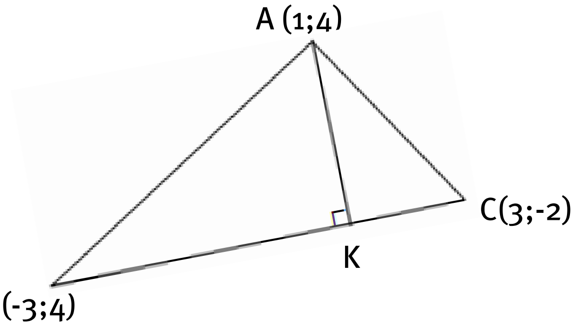
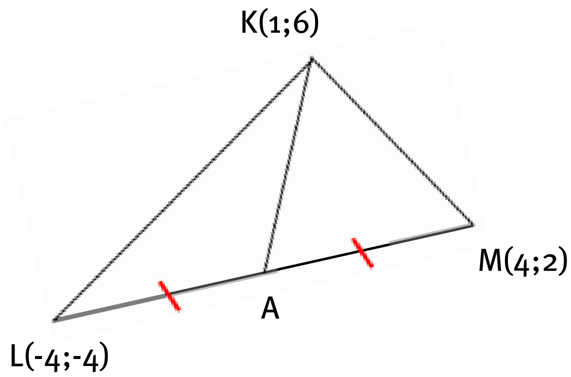
Analytical geometry

REVISION OF CONCEPTS FROM PREVIOUS GRADES

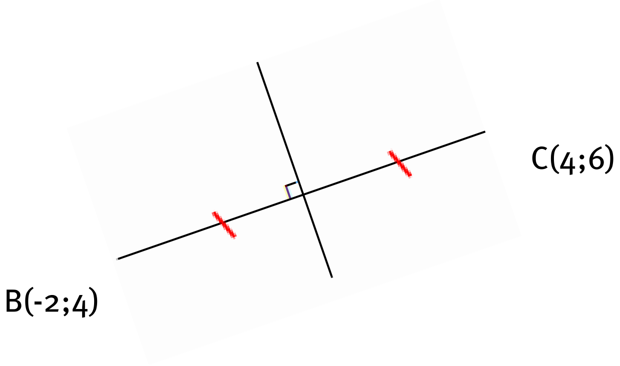
CONCEPT	FORMULA / METHOD
Distance between two points $A(x_1; y_1)$ and $B(x_2; y_2)$	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Coordinates of midpoint	$\left(\frac{x_1 + x_2}{2}; \frac{y_1 + y_2}{2}\right)$
Average gradient between two points $A(x_1; y_1)$ and $B(x_2; y_2)$ **Gradient of straight line through A and B**	$m = \frac{y_2 - y_1}{x_2 - x_1}$ Or when given the angle of inclination, θ, use $m = tan\theta$
Equation of a straight line	$y = mx + c$ Or $y - y_1 = m(x - x_1)$
Angle of inclination, θ **NB: Angle between line and** POSITIVE **x –axis**	$m = tan\theta$  If $m > 0(+)$, then θ is an acute angle (smaller than 90°)  If $m < 0(-)$, then θ is an obtuse angle (bigger than 90°but)
To prove that points A, B and C are collinear (i.e. arranged in a straight line)	Prove that $m_{AB} = m_{BC}$ Or $m_{AB} = m_{AC}$ Or $m_{AC} = m_{BC}$
Parallel lines	Two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are parallel if $m_1 = m_2$
Perpendicular lines	Two lines $y = m_1x + c_1$ and $y = m_2x + c_2$ are perpendicular if $m_1 \times m_2 = -1$

Chapter 9

Analytical geometry

OTHER DEFINITIONS/CONCEPTS YOU HAVE TO KNOW	
DEFINITION	**EXAMPLES**
Altitude of a triangle = line from one vertex perpendicular to opposite side	 To determine the equation of altitude AK: • Determine gradient of BC • Determine gradient of AK • Determine equation of AK (substitute A) $m_{BC} = \frac{-2-4}{3-(-3)} = -1$ But BC⊥AK, so $m_{AK} = 1$ Substitute point A(1;4): $y - 4 = -1(x - 1)$ Equation of AK: $y = -x + 5$
Median = line joining vertex of triangle to midpoint of opposite side	 To determine the equation of median KA: • Determine coordinates of midpoint A • Determine gradient of KA • Determine equation of KA $A\left(\frac{-4+4}{2};\frac{-4+2}{2}\right)$ $A(0;-1)$ $m_{KA} = \frac{6-(-1)}{1-0} = 7$ $y - 6 = 7(x - 1)$ $y = 7x - 1$

Perpendicular bisector = the line through the midpoint of a line and perpendicular to that line



To determine equation of perpendicular bisector of BC:

- Determine gradient of BC
- Determine gradient of bisector
- Determine equation of bisector

$$m_{BC} = \frac{6-4}{4-(-2)} = \frac{1}{3}$$

Product of gradients must be -1:

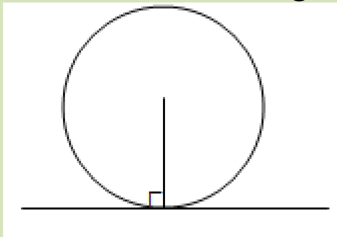
$$m_{perp\ bisector} = -3$$

$$y - 6 = -3(x - 4)$$
$$y = -3x + 18$$

THE CIRCLE : $(x-a)^2+(y-b)^2=r^2$

(centre-radius form)

SUMMARY ON CIRCLES	
Equation of circle with radius r and centre at the origin	$x^2+y^2=r^2$
Equation of circle with radius r and centre $(a;b)$	$(x-a)^2+(y-b)^2=r^2$
To determine radius and centre of circle when given equation	Example A: Determine the radius and centre of the circle with equation $(x+1)^2+(y-3)^2=16$ $(x+1)^2+(y-3)^2=16$ can be written as $(x-(-1))^2+(y-(3))^2=16$ Centre: $(-1;3)$ Radius = $\sqrt{16}=4$ Example B: Determine the radius and centre of the circle with equation $x^2+y^2+4x+6y-10=0$ We are going to use **COMPLETION OF THE SQUARE** • Constant term to RHS; group x- and y-terms $x^2+4x+\cdots+y^2+6y+\cdots=10$ • Complete square for x and y - add $\left(\frac{1}{2}\times coefficient\right)^2$ $x^2+4x+\color{red}{4}+y^2+6y+\color{blue}{9}=10+\color{red}{4}+\color{blue}{9}$ • Write in centre-radius form $(x+2)^2+(y+3)^2=23$ Centre: $(-2;-3)$ Radius = $\sqrt{23}$

Equation of tangent to circle at given point	**NB: Radius is PERPENDICULAR to tangent**  Determine the equation of the tangent to the circle $(\boldsymbol{x}+\mathbf{1})^2+(\boldsymbol{y}-\mathbf{3})^2=\mathbf{16}$ through the point $(\mathbf{2};\mathbf{5})$. The steps are: • Determine centre of circle • Determine gradient of RADIUS • Determine gradient of TANGENT: Remember: $\boldsymbol{m_{rad}}\times\boldsymbol{m_{tan}}=-\mathbf{1}$ • Determine equation of tangent by substituting point of tangency The centre of circle $(\boldsymbol{x}+\mathbf{1})^2+(\boldsymbol{y}-\mathbf{3})^2=\mathbf{16}$ is $(-\mathbf{2};\mathbf{3})$. Radius joins centre $(-\mathbf{1};\mathbf{3})$ with point of tangency $(\mathbf{2};\mathbf{5})$. $\boldsymbol{m_{rad}}=\frac{5-3}{2-(-2)}=\frac{2}{4}=\frac{1}{2}$ $\therefore \boldsymbol{m_{tangent}}=-2$ Equation of tangent: $\boldsymbol{y}-\mathbf{5}=-\mathbf{2}(\boldsymbol{x}-\mathbf{2})$ $\boldsymbol{y}=-\mathbf{2}\boldsymbol{x}+\mathbf{9}$

Mixed Exercise on Analytical Geometry

1 A $(-2;1)$, B$(p;-4)$, C$(5;0)$ and D $(3;2)$ are the vertices of trapezium ABCD in a Cartesian plane with $AB \parallel CD$.

a Show that $p = 3$.

b Calculate AB:CD in simplest form.

c If N $(x;y)$ is on AB and NBCD is a parallelogram, determine the coordinates of N.

d Determine the equation of the line passing through B and D.

e What is the angle of inclination of line BD?

f Calculate the area of parallelogram NBCD.

g R $(-1;\ q)$, A and C are collinear. Calculate the value of q.

2 $x^2 + 4x + y^2 + 2y - 8 = 0$ is the equation of a circle with centre M in a Cartesian plane.

a Prove that the circle passes through the point N$(1; -3)$

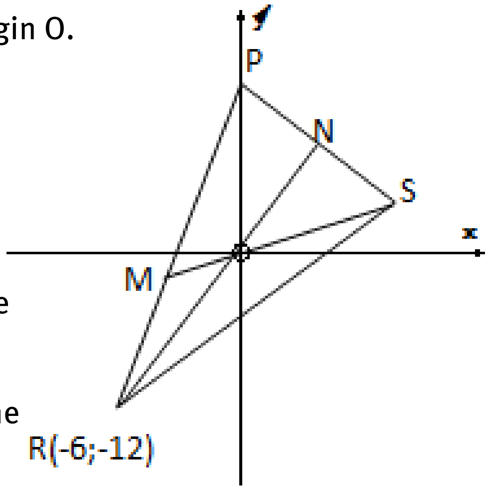
b Determine the equation of PN, the tangent to the circle at N.

c Calculate θ, the angle of inclination of the tangent, rounded off to <u>one</u> decimal place.

d Determine the coordinates of the point where the tangent in 2 b intersects the x-axis.

e Calculate the coordinates of the point(s) where the circle with centre M cuts the y-axis.

3 In the diagram, P, R and S are vertices of ΔPRS . P is a point on the y-axis. The coordinates of R is (-6; -12). The equation of PR is $3x - y + 6 = 0$.
The median SM and the altitude RN intersect at the origin O.



a Calculate the gradient of RO.

b Calculate the gradient of PS.

c Determine the equation of PS.

d Calculate the inclination of PS rounded of to one decimal digit.

e If the coordinates of N are $\left(2n; 3\frac{3}{5}+n\right)$, determine the value of n.

f Calculate the coordinates of S.

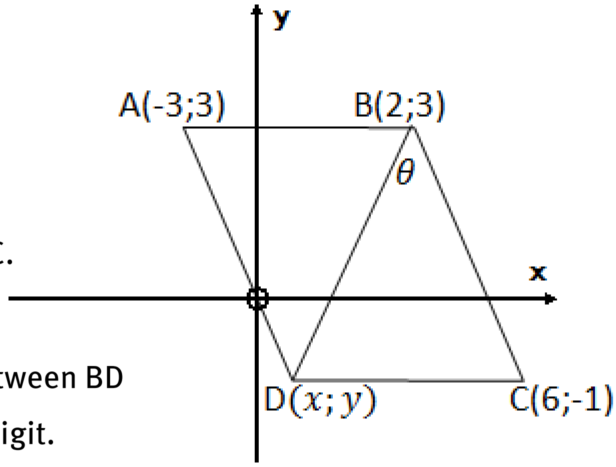
4 The equation of a circle is $x^2 + y^2 + 4x - 2y - 4 = 0$.

a Determine the coordinates of M, the centre of the circle, as well as the length of the radius.

b Calculate the value of p if N$(p;\ 1)$ with $p > 0$, is a point on the circle.

c Write down the equation of the tangent to the circle at N.

5 A (-3; 3), B(2; 3), C(6; - 1) and D$(x;\ y)$ are the vertices of quadrilateral ABCD in a Cartesian plane.



a Determine the equation AD.

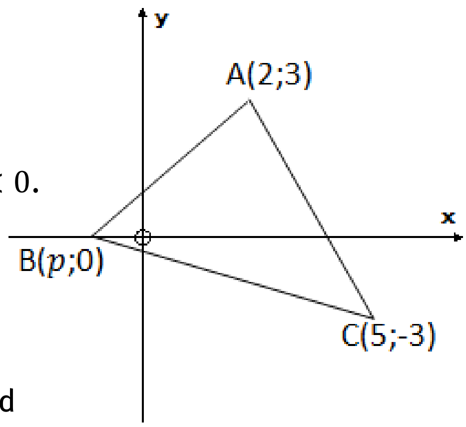
b Prove that the coordinates of D are $\left(\frac{3}{2}; -\frac{3}{2}\right)$ if D is equidistant from B and C.

c Determine the equation of BD.

d Determine the size of θ, the angle between BD and BC, rounded off to one decimal digit.

e Calculate the area of ΔBDC rounded off to the nearest square unit.

6 In the diagram, points A(2; 3), B$(p; 0)$ and C(5; - 3) are the vertices of ΔABC in a Cartesian plane. AC cuts the x-axis at D.



a Calculate the coordinates of D.

b Calculate the value of p if BC = AC and $p < 0$.

c Determine the angle of inclination of straight line AC, rounded off to one decimal place.

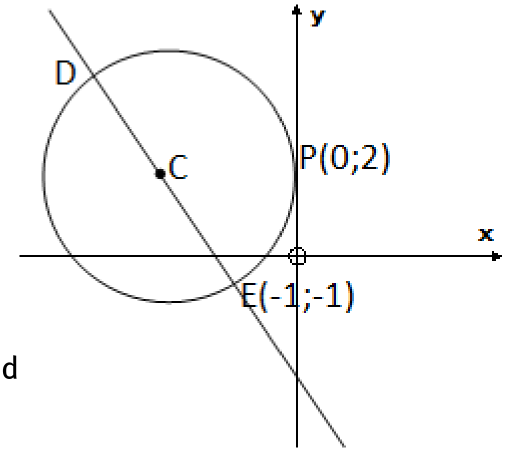
d If $p = -1$, calculate the size of $\hat{A}$, rounded off to one decimal digit.

7 In the Cartesian plane the equation of a circle with centre M is given by:

$$x^2 + y^2 + 6y - 7 = 0$$

Determine, by calculation, whether the straight line $y = x + 1$ is a tangent to the circle, or not.

8 In the diagram, centre C of the circle lies on the straight line $3x + 4y + 7 = 0$. The straight line cuts the circle at D and E$(-1; -1)$. The circle touches the y-axis at P$(0;\ 2)$.



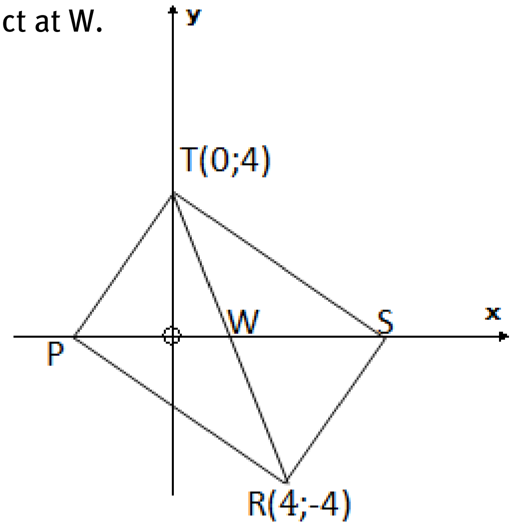
a Determine the equation of the circle in the form: $(x - n)^2 + (y - q)^2 = r^2$

b Determine the length of diameter DE.

c Determine the equation of the perpendicular bisector of PE.

d Show that the perpendicular bisector of PE and straight line DE intersect at C.

9 In the diagram, P, R$(4; -4)$, S and T (0; 4) are the vertices of a rectangle. P and S lie on the x – axis. The diagonals intersect at W.



a Show that the coordinates of S are $(2 + 2\sqrt{5}; 0)$.

b Determine the gradient of TS rounded off to two decimals.

c Calculate $R\hat{T}S$ rounded off to two decimals.

10

a Show that the equation of the tangent to the circle $x^2 + y^2 - 4x + 6y + 3 = 0$ at the point (5; -2) is $y = -3x + 13$

b If T$(x;\ y)$ is a point on the tangent in 10.a, such that its distance from the centre of the circle is $\sqrt{20}$ units, determine the values of x and y.

Chapter 10

Euclidean geometry

Overview

Chapter10 Page 236 Euclidean geometry	Unit 1 Page 244	
	Proportionality in triangles	• Ratio • Theorem 1
	Unit 2 Page 250	
	Similarity in triangles	• Theorem 2 • Theorem 3
	Unit 3 Page 256	
	Theorem of Pythagoras	• Prove of Theorem of Pythagoras

REMEMBER YOUR STUDY APPROACH SHOULD BE:

1 Work through all examples in this chapter of your Learner's Book.

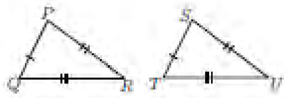
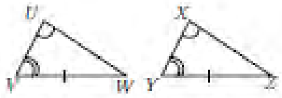
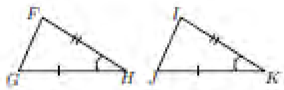
2 Work through the notes in this chapter of the study guide.

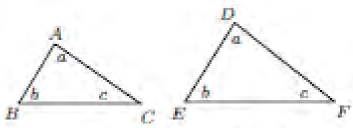
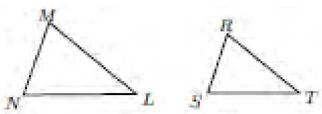
3 Do the exercises at the end of the chapter in the Learner's Book.

4 Do the mixed exercises at the end of this chapter in the study guide.

The only way to STUDY Maths is to DO Maths!

REVISION OF GEOMETRY FROM PREVIOUS YEARS

CONGRUENCY	
SSS	 $\triangle PQR \equiv \triangle STU$
AAS	 $\triangle UVW \equiv \triangle XYZ$
SAS (included angle)	 $\triangle FGH \equiv \triangle IJK$
RHS	 $\triangle ABC \equiv \triangle DEF$

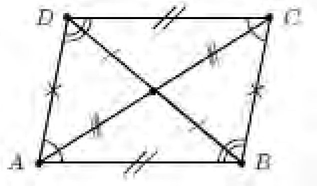
SIMILARITY	
AAA	 $\hat{A} = \hat{D},\ \hat{B} = \hat{E},\ \hat{C} = \hat{F}$ $\therefore \triangle ABC \mathbin{\|\|\|} \triangle DEF$
SSS	 $\frac{MN}{RS} = \frac{ML}{RT} = \frac{NL}{ST}$ $\therefore \triangle MNL \mathbin{\|\|\|} \triangle RST$

Euclidean geometry

PROPERTIES OF SPECIAL QUADRILATERALS

PARALLELOGRAM

- Both pairs of opposite sides are parallel
- Both pairs of opposite side are equal
- Both pairs of opposite angles are equal
- Diagonals bisect each other

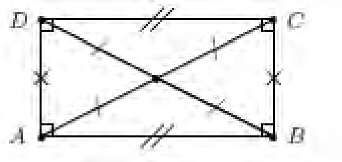


RECTANGLE

All properties of **parallelogram**

Plus:

- Both diagonals are equal in length
- All interior angles are equal to 90°

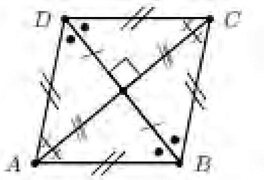


RHOMBUS

All properties of **parallelogram**

Plus:

- All **sides** are **equal**
- Diagonals bisect each other **perpendicularly**
- Diagonals **bisect** interior **angles**

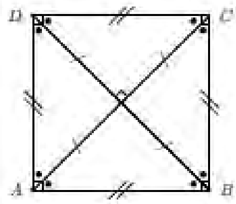


SQUARE

All properties of a **rhombus**

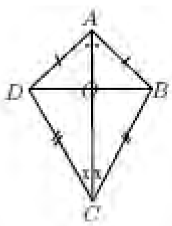
Plus:

- All interior angles are 90°
- Diagonals are equal in length



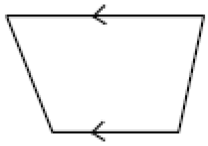
KITE

- Two pairs of adjacent sides are equal
- Diagonal between equal sides bisects other diagonal
- One pair of opposite angles are equal (unequal sides)
- Diagonal between equal sides bisects interior angles (is axis of symmetry)
- Diagonals intersect perpendicularly



TRAPEZIUM

- One pair of opposite sides are parallel



HOW TO PROVE THAT A QUADRILATERAL IS A PARALLELOGRAM

Prove any ONE of the following (most often by congruency):

- Prove that both pairs of opposite **sides** are **parallel**
- Prove that both pairs of opposite **sides** are **equal**
- Prove that both pairs of opposite **angles** are **equal**
- Prove that the **diagonals bisect** each other

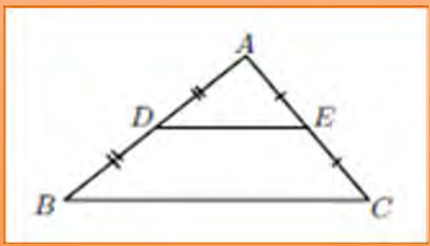
HOW TO PROVE THAT A PARALLLELOGRAM IS A RHOMBUS

Prove ONE of the following:

- Prove that the diagonals bisect each other **perpendicularly**
- Prove that any two **adjacent sides** are **equal** in length

MIDPOINT THEOREM

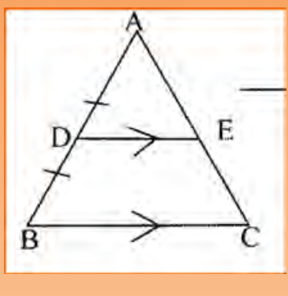
The line segment joining the midpoints of two sides of a triangle, is parallel to the 3rd side of the triangle and half the length of that side.



If AD = DB and AE = EC, then DE ∥ BC and DE = $\frac{1}{2}$BC

CONVERSE OF MIDPOINT THEOREM

If a line is drawn from the midpoint of one side of a triangle parallel to another side, that line will bisect the 3rd side and will be half the length of the side it is parallel to.



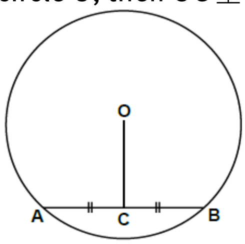
If AD = DB and DE ∥ BC, then AE = EC and DE = $\frac{1}{2}$BC.

Chapter 10
Euclidean geometry

REVISION OF CIRCLE GEOMETRY (FROM GRADE 11)

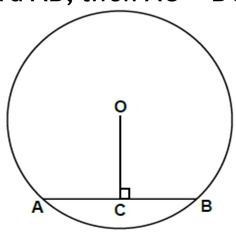
<u>Theorem 1</u>

If AC = CB in circle O, then OC ⊥ AB.



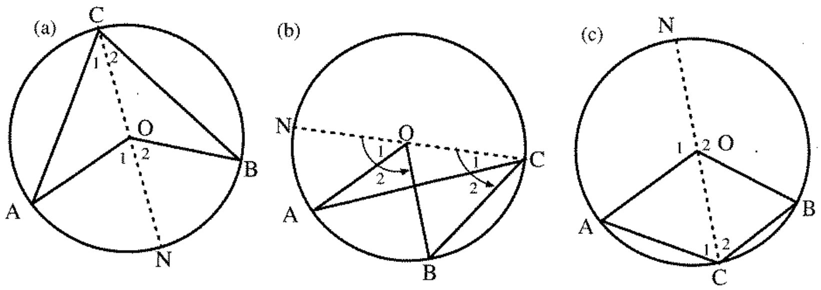
<u>Converse of Theorem 1</u>

If OC ⊥ chord AB, then AC = BC.



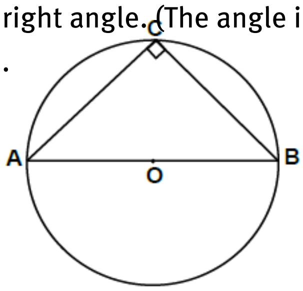
<u>Theorem 2</u>

The angle at the centre of a circle subtended by an arc/a chord is double the angle at the circumference subtended by the same arc/chord. $A\hat{O}B = 2 \times A\hat{C}B$



<u>Theorem 3</u>

The angle on the circumference subtended by the diameter, is a right angle. (The angle in a semi-circle is $90°$).

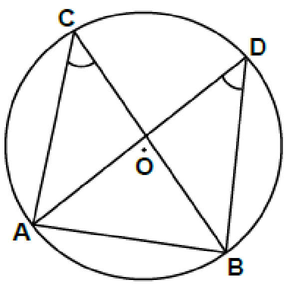


<u>Converse of Theorem 3</u>

If $\hat{C} = 90°$, then AB is the diameter of the circle.

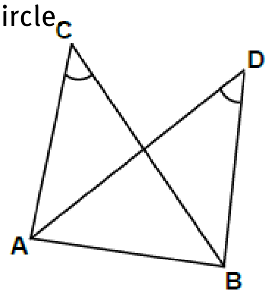
<u>Theorem 4</u>

The angles on the circumference of a circle, subtended by the same arc or chord, are equal.



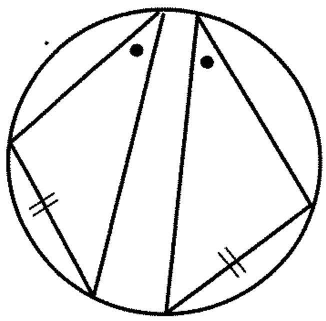
<u>Converse of Theorem 4</u>

If a line segment subtends equal angles at two other points, then these four points lie on the circumference of a circle.

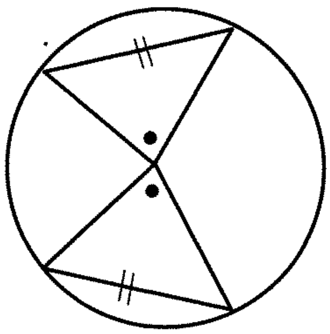


Corollaries of Theorem 4

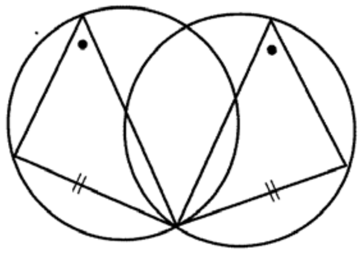
Equal chords subtend equal angles at the circumference of the circle.



Equal chords subtend equal angles at the centre of the circle.



Equal chords of equal circles subtend equal angles at the circumference.

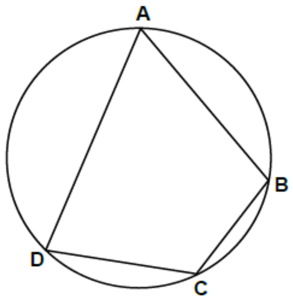


Theorem 5

The opposite angles of a cyclic quadrilateral are supplementary.

$\hat{A} + \hat{C} = 180°$

$\hat{B} + \hat{D} = 180°$

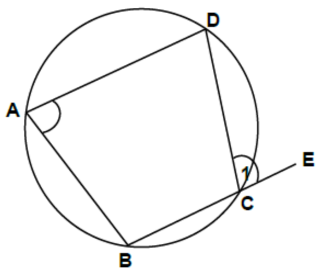


Converse of Theorem 5

If the opposite angles of a quadrilateral are supplementary, then it is a cyclic quadrilateral.

Theorem 6

The exterior angle of a cyclic quadrilateral is equal to the opposite interior angle.

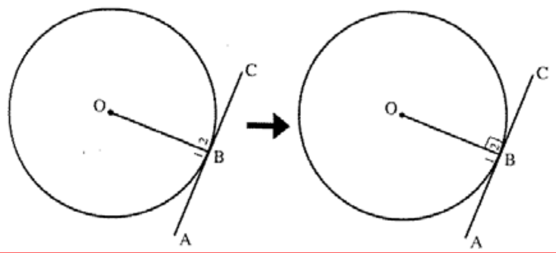


Converse of Theorem 6

If the exterior angle of a quadrilateral is equal to the opposite interior angle, then it is a cyclic quadrilateral.

Theorem 7

The tangent to a circle is perpendicular to the radius at the point of tangency.

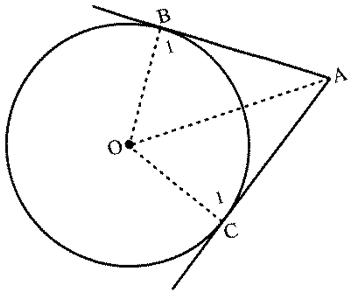


Converse of Theorem 7

If a line is drawn perpendicularly to the radius through the point where the radius meets the circle, then this line is a tangent to the circle.

Theorem 8

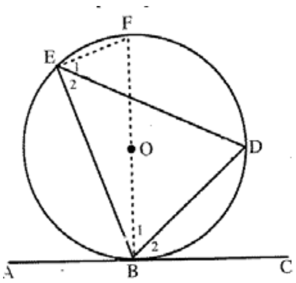
If two tangents are drawn from the same point outside a circle, then they are equal in length.



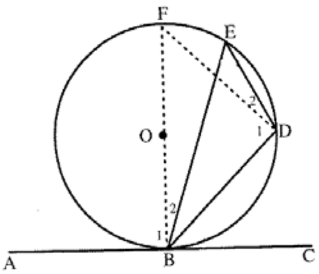
Theorem 9 (Tan chord theorem)

The angle between the tangent to a circle and a chord drawn from the point of tangency, is equal to the angle in the opposite circle segment.

Acute angle



Obtuse angle



Converse of Theorem 9

If a line is drawn through the endpoint of a chord to form an angle which is equal to the angle in the opposite segment, then this line is a tangent.

THREE WAYS TO PROVE THAT A QUADRILATERAL IS A CYCLIC QUADRILATERAL

Prove that:

- one pair of opposite angles are supplementary
- the exterior angle is equal to the opposite interior angle
- two angles subtended by a line segment at two other vertices of the quadrilateral, are equal.

Example 1

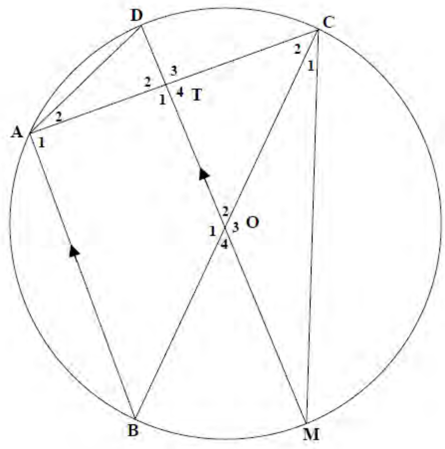
In the diagram alongside‹ O is the centre of circle DABMC.

BC and DM are diameters.

AC and DM intersect at T.

OT =3DT

AB∥DM



a Prove that T is the midpoint of AC.

b Determine the length of MC in terms of DT.

c Express $\widehat{D}$ in terms of $\hat{O}_2$.

Solution:

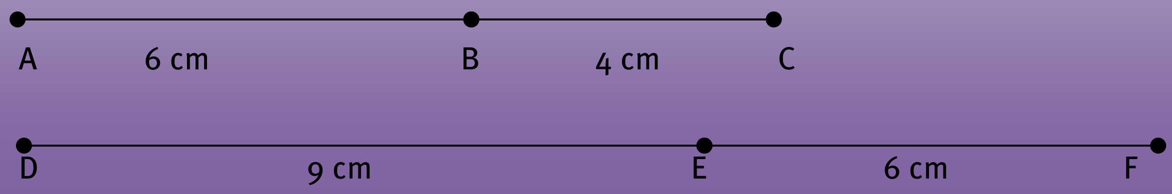
a $\hat{A}_1 = 90°$ $\angle$ in semi $\odot$

$\hat{T}_1 = 90°$ int. $\angle$s suppl

Chapter 10

Euclidean geometry

REVISING THE CONCEPT OF PROPORTIONALITY



AB : BC = 6 : 4 = **3 : 2**

DE : EF = 9 : 6 = **3 : 2**

Although, AB : BC = DE : EF it does **NOT** mean that AB = DE, AC = DF or BC = EF.

GRADE 12 GEOMETRY

Theorem 1

A line drawn parallel to one side of a triangle that intersects the other two sides, will divide the other two sides proportionally.

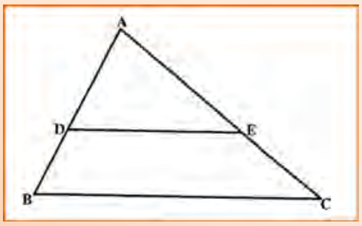
If DE ∥ BC then $\frac{AD}{DB} = \frac{AE}{EC}$

or AD : DB = AE : EC

Converse of Theorem 1

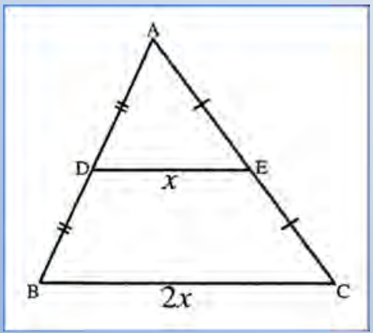
If a line divides two sides of a triangle proportionally, then the line is parallel to the third side of the triangle.

If $\frac{AD}{DB} = \frac{AE}{EC}$ then DE ∥ BC.



Theorem 2 (Midpoint Theorem)
(Special case of Theorem 1)
The line segment joining the midpoints of two sides of a triangle, is parallel to the 3rd side of the triangle and half the length of that side.

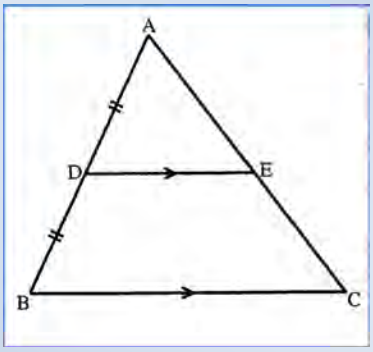
If AD = DB and AE = EC, then DE ∥ BC and DE = $\frac{1}{2}$BC.



Converse of Theorem 2

If a line is drawn from the midpoint of one side of a triangle parallel to another side, that line will bisect the 3rd side and will be half the length of the side it is parallel to.

If AD = DB and DE ∥ BC, then AE = EC and DE = $\frac{1}{2}$BC.



Theorem 3

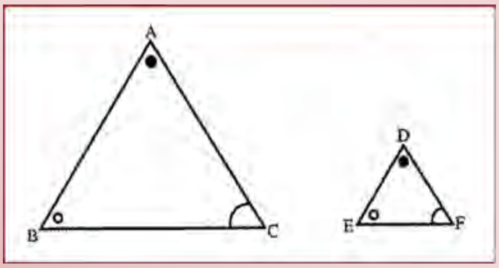
The corresponding sides of two equiangular proportional, triangles are proportional.

If $\Delta ABC\ |||\ \Delta DEF$ then $\frac{AB}{DE}=\frac{BC}{EF}=\frac{AC}{DF}$

Converse of Theorem 3

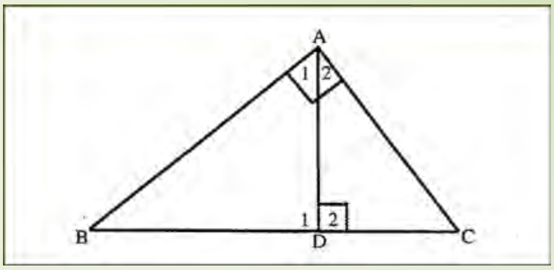
If the sides of two triangles are then the triangles are equiangular.

If $\frac{AB}{DE}=\frac{BC}{EF}=\frac{AC}{DF}$ then $\Delta ABC\ |||\Delta DEF$



Theorem 4

The perpendicular drawn from the vertex of the right angle of a right-angled triangle, divides the triangle in two triangles which are similar to each other and similar to the original triangle.



Corollaries of Theorem 4

$\Delta ABC|||\Delta DBA$

$\therefore \frac{AB}{DB} = \frac{BC}{BA} = \frac{AC}{DA}$

$\therefore AB^2 = BD.BC$

$\Delta ABC|||\Delta DAC$

$\therefore \frac{AB}{DA} = \frac{BC}{AC} = \frac{AC}{DC}$

$\therefore AC^2 = CD.CB$

$\Delta DBA|||\Delta DAC$

$\therefore \frac{DB}{DA} = \frac{BA}{AC} = \frac{DA}{DC}$

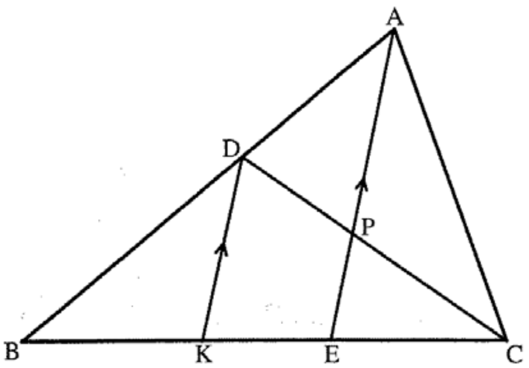
$\therefore AD^2 = BD.DC$

Theorem 5 (The Theorem of Pythagoras)

Using the corollaries of Theorem 4, it can be proven that:

$$BC^2 = AB^2 + AC^2$$

Example



Given: $AD:DB = 2:3$ and $BE = \frac{4}{3}EC$.

Instruction: Determine the ratio of $CP:PD$.

Solution:

In ΔABE $\quad \frac{BE}{KE} = \frac{5}{2} \quad \therefore BE = \frac{5}{2}KE$

But it was given that $BE = \frac{4}{3}EC$

$\therefore \frac{4}{3}EC = \frac{5}{2}KE$

$\frac{EC}{KE} = \frac{5}{2} \div \frac{4}{3} = \frac{15}{8}$

In ΔCDK $\quad \frac{CP}{PD} = \frac{CE}{EK} = \frac{15}{8}$

$\therefore CP:PD = 15:8$

TIPS TO SOLVE A GEOMETRY PROBLEM

- **READ-READ-READ** the information next to the diagram thoroughly
- TRANSFER all given information on the **DIAGRAM**
- Look for **KEYWORDS**, e.g.

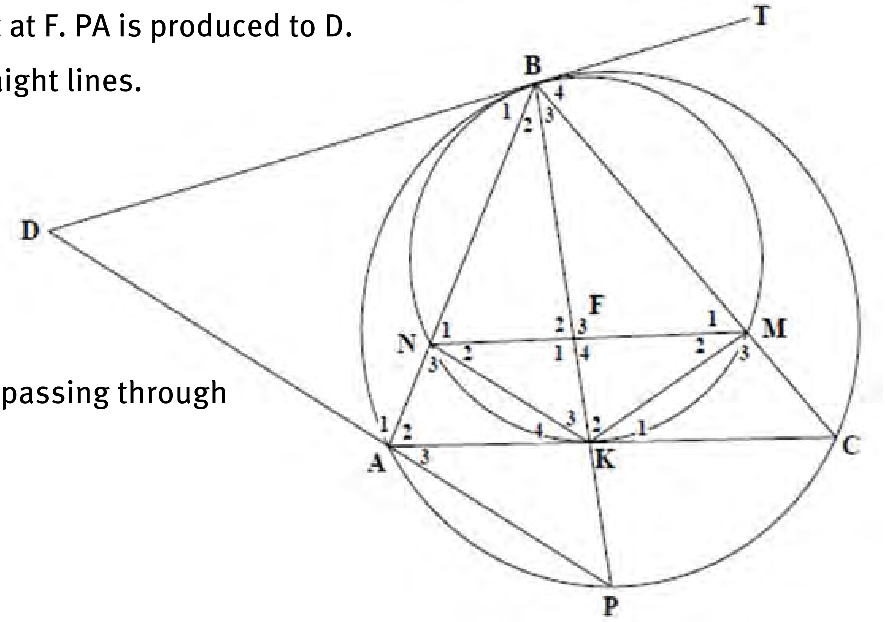
  TANGENT: What do the theorems say about tangents?

  CYCLIC QUADRILATERAL: What are the properties of a cyclic quad?
- Set yourself **"SECONDARY" GOALS**, e.g.
  - To prove that two sides of triangle are equal (primary goal), first prove that there are two equal angles (secondary goal)
  - To prove that a line is a tangent, the secondary goal can be to prove that the line is perpendicular to a radius
- For questions like: Prove that $\hat{A}_1 = \hat{C}_2$. Start with ONE PART. Move to the OTHER PART **step-by-step** stating reasons.

  E.g. $\hat{A}_1 = \hat{A}_2$; $\hat{A}_2 = \hat{C}_1$; $\hat{C}_1 = \hat{C}_2$; $\therefore \hat{A}_1 = \hat{C}_2$

Mixed Exercise on Euclidian Geometry

1 In the diagram, TBD is a tangent to circles BAPC and BNKM at B.
AKC is a chord of the larger circle and is also a tangent to the smaller circle at K.
Chords MN and BK intersect at F. PA is produced to D.
BMC, BNA and BFKP are straight lines.
Prove that:



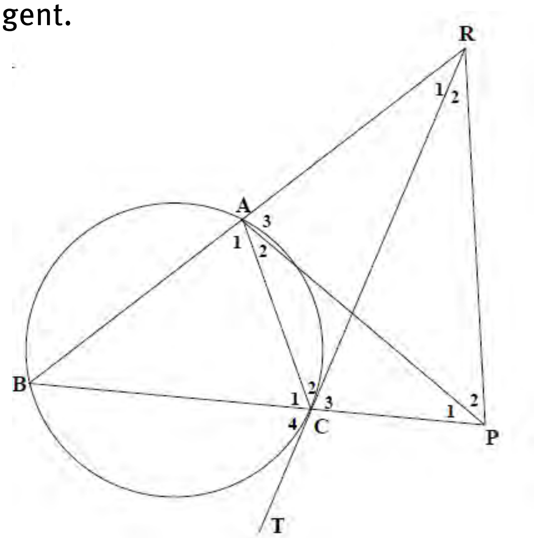
a MN ∥ CA

b ΔKMN is isosceles

c $\frac{BK}{KP} = \frac{BM}{MC}$

d DA is a tangent to the circle passing through points A, B and K.

2 In the diagram below, chord BA and tangent TC of circle ABC are produced to meet at R. BC is produced to P with RC=RP. AP is not a tangent.



Prove that:

a ACPR is a cyclic quadrilateral.

b $\Delta CBA|||\Delta RPA$

c $RC = \frac{CB.RA}{AC}$

d $RB.AC = RC.CB$

e Hence prove that $RC^2 = RA.RB$

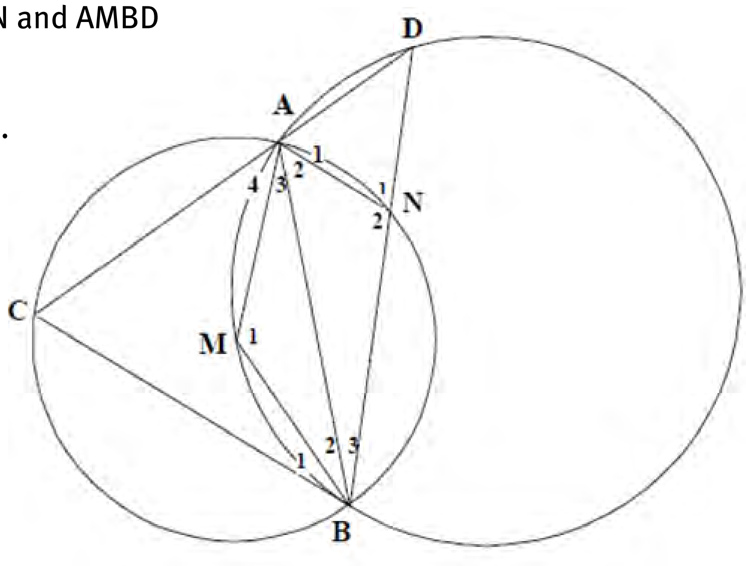
3 In the diagram alongside, circles ACBN and AMBD Intersect at A and B.

CB is a tangent to the larger circle at B.

M is the centre of the smaller circle.

CAD and BND are straight lines.

Let $\hat{A}_3 = x$



a Determine the size of $\widehat{D}$ in terms of x.

b Prove that:

i CB ∥ AN

ii AB is a tangent to circle ADN.

4 In the diagram below, O is the centre of circle ABCD.

DC is extended to meet circle BODE at point E.

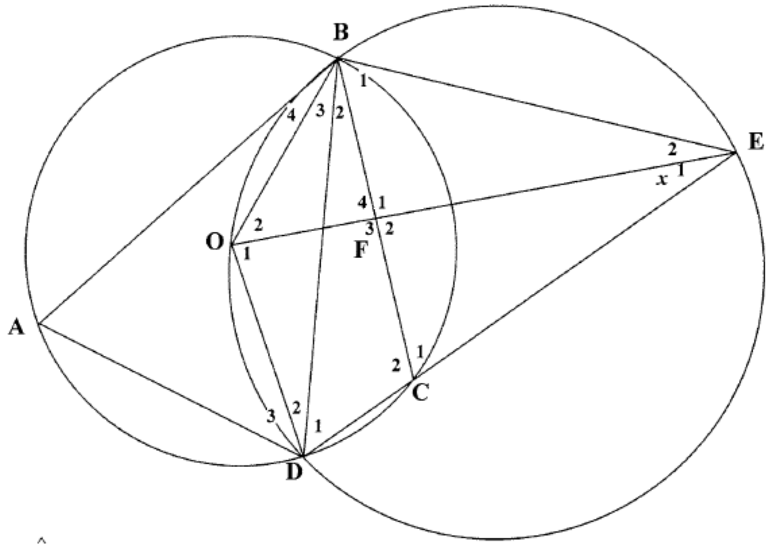
OE cuts BC at F. Let $\hat{E}_1 = x$.

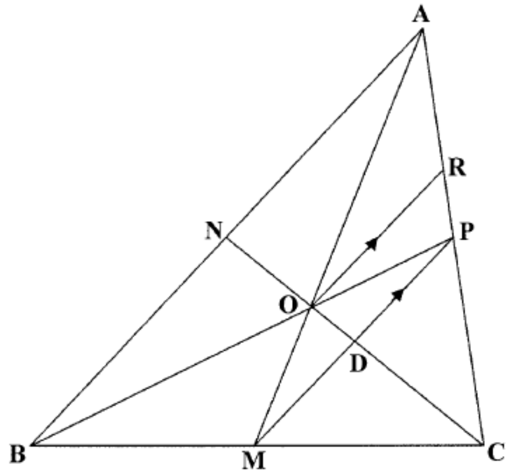
a Determine $\hat{A}$ in terms of x.

b Prove that:

i BE=EC

ii BE is NOT a tangent to circle ABCD.





5 In the diagram alongside, medians AM and CN of ΔABC intersect at O.

BO is produced to meet AC at P.

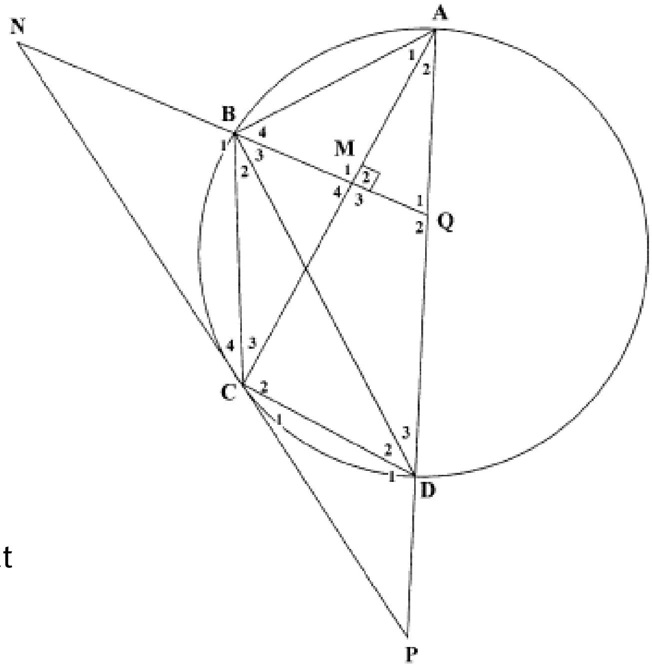
MP and CN intersect in D.

OR∥MP with R on AC.

a Calculate, giving reasons, the numerical value of $\frac{ND}{NC}$.

b Use $AO: AM = 2:3$, to calculate the numerical value of $\frac{RP}{PC}$.

6 In the diagram, AD is the diameter of circle ABCD.
AD is extended to meet tangent NCP in P.
Straight line NB is extended to Q and intersect AC in M with Q on straight line ADP.
AC⊥NQ at M.



a Prove that NQ∥CD.

b Prove that ANCQ is a cyclic quadrilateral.

c i Prove that $\Delta PCD|||\Delta PAC$.

ii Hence, complete: $PC^2 = \cdots$

d Prove that $BC^2 = CD.NB$

e If it is further given that PC=MC, prove that

$$1 - \frac{BM^2}{BC^2} = \frac{AP.DP}{CD.NB}$$

Overview

Chapter 11 Page 262 Statistics: regression and correlation	**Unit 1 Page 266**	
	Symmetrical and skewed data	• Symmetrical data • Skewed data
	Unit 2 Page 270	
	Scatter plots and correlation	• Bivariate data • Correlation • Examples of scatter plots and correlation • Drawing scatter plots • The least squares method • The correlation coefficient • Using a calculator to find the regression line

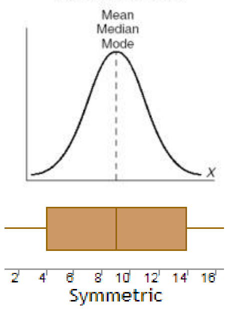
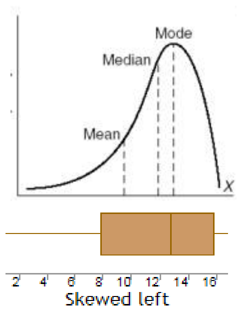
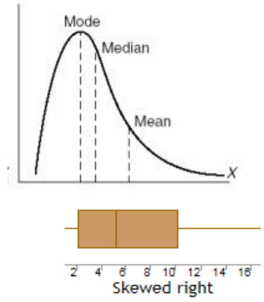
REMEMBER YOUR STUDY APPROACH SHOULD BE:

1 Work through all examples in this chapter of your text book.
2 Work through the notes in this chapter of the study guide.
3 Do the exercises at the end of the chapter in the text book.
4 Do the mixed exercises at the end of this chapter in the study guide.

The only way to STUDY Maths is to DO Maths!

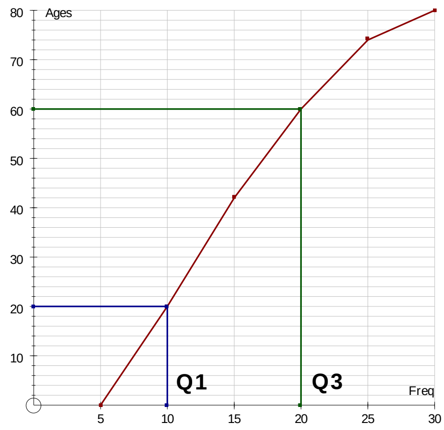
Statistics: regression and correlation

	UNGROUPED DATA	GROUPED DATA
Measures of central tendency	Mode = most frequent number	Modal class = interval with highest frequency
	Mean $= \frac{Sum\ of\ the\ values}{Number\ of\ values}$	Estimated mean $= \frac{\sum x_i \times f_i}{\sum f_i}$ where x_i = midpoint of class i and f_i = frequency of class i
	NB: Data has to be arranged in ascending order Q_2, Median = Middle value (for an odd number of values) Or $\frac{Sum\ of\ two\ middle\ values}{2}$ (for an even number of values)	Median class interval = class/interval in which middle value lies **Position** of $Q_2 = \frac{(n+1)}{2}$
Measures of dispersion (indicates spread of data)	Percentiles (divide data into 100 equal parts) E.g. the position of $P_{30} = \frac{30}{100}(n+1)$	
	Q_1, Lower quartile = Middle value of all the values **below** the median (excluding median)	**Position** of $Q_1 = \frac{(n+1)}{4}$
	Q_3, Upper quartile = Middle value of all the values **above** the median (excluding median)	**Position** of $Q_3 = \frac{3(n+1)}{4}$
	Range = Maximum – Minimum	
	Inter quartile range (IQR) $= Q_3 - Q_1$	
	Semi Inter quartile range $= \frac{Q_3 - Q_1}{2}$	
	Five point summary (used to draw box-and-whisker diagram): Min, Q_1, Median, Q_3, Max	

DISTRIBUTION OF DATA		
SYMMETRICAL DISTRIBUTION	**ASYMMETRICAL** DISTRIBUTIONS	
NORMAL DISTRIBUTION	**NEGATIVELY SKEWED**	**POSITIVELY SKEWED**
$\boldsymbol{mean = median = mode}$	$mean - median < 0$	$mean - median > 0$
		

OGIVE

Ogive = cumulative frequency graph



NB: When drawing the ogive:

- plot the (upper class boundary ; cumulative frequency)
- the graph has to be grounded
- the shape of the graph has to be smooth rather than consist of "connected dots"

THE OGIVE CAN BE USED TO DETERMINE THE MEDIAN AND QUARTILES.

MEASURES OF DISPERSION AROUND THE MEAN

VARIANCE σ^2

Variance, σ^2, is an indication of how far each value in the data set is from the mean, $\bar{x}$.

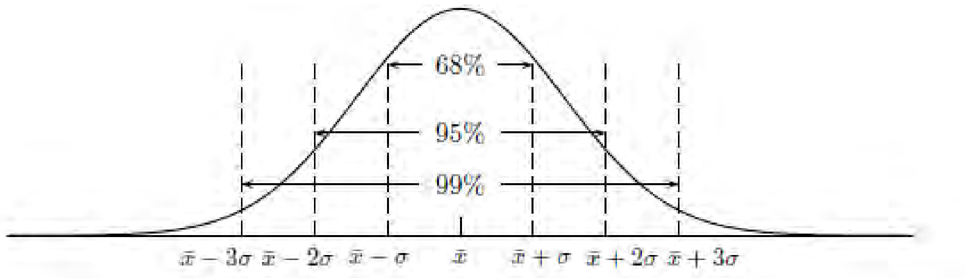
$$\sigma^2 = \frac{\sum(x_i - \bar{x})^2}{n} \quad \text{(for population)}$$

STANDARD DEVIATION σ

Standard deviation (SD), σ: SD $= \sqrt{variance}$

The larger the standard deviation, the larger the deviation from the mean would be.

A normal distribution is shown below:



USING A TABLE TO CALCULATE VARIANCE AND STANDARD DEVIATION

UNGROUPED DATA

First calculate the **mean,**$\bar{x}$ then the following columns.

DATA VALUES, x	$(x-\bar{x})$	$(x-\bar{x})^2$

Calculate the total of this column,$\sum(x-\bar{x})^2$

Variance $=\frac{\sum(x-\bar{x})^2}{n}$

Standard variance $=\sqrt{variance}$

GROUPED DATA

First calculate the **estimated mean,** $\bar{x}=\frac{\sum f\times m}{\sum f}$

Class Interval	Frequency f	Midpoint m	$f\times m$	$m-\bar{x}$	$(m-\bar{x})^2$	$f\times(m-\bar{x})^2$

Calculate the total of this column

Variance $=\frac{\sum f(x-\bar{x})^2}{n}$

Standard variance $=\sqrt{variance}$

USING CASIO fx-82ZA PLUS CALCULATOR TO CALCULATE STANDARD DEVIATION

MODE

2 : STAT

1 : 1 – VAR

Enter the data points: Push = after each data point

AC

SHIFT STAT (above the 1 button)

4 : VAR

3: σxn

To switch on the **frequency column** when calculating the SD for a frequency table, first do the following:

Shift Setup; Down arrow (on big REPLAY button); 3: STAT; 2: ON

To clear screen: MODE 1: COMP

DETERMINING OUTLIERS

Inter quartile range, IQR $= Q_3 - Q_1$

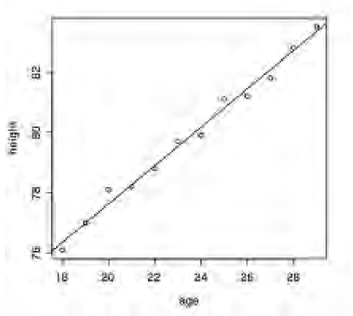
An outlier is identified if it is:

- Less than $Q_1 - IQR \times 1{,}5$ or
- Larger than $Q_3 + IQR \times 1{,}5$

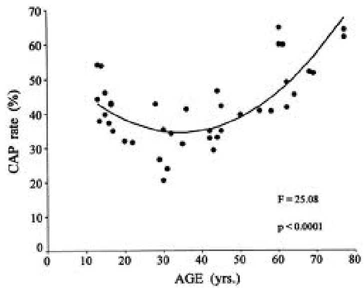
SCATTER DIAGRAMS (SCATTER PLOTS) FOR BIVARIATE DATA

Scatter diagrams are used to graphically determine whether there is an association between two variables.

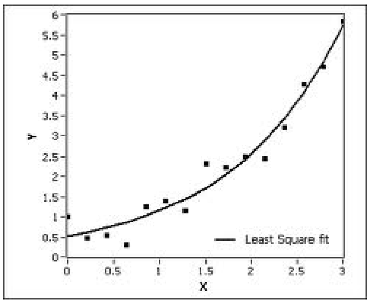
By investigation one can determine which of the following curves (regression functions) would best fit the diagram:



Linear (straight line)



Quadratic (parabola)



Exponential function

USING A CALCULATOR TO DETERMINE THE EQUATION OF THE REGRESSION LINE (LEAST SQUARES REGRESSION LINE)

The standard form of a straight line equation is: $y = mx + c$

where m is the gradient and c is the y-intercept.

NB: On the calculator the regression line is determined in the form: $\boldsymbol{y = A + Bx}$

(In this form $B =$ the gradient of the line and A = the y-intercept)

On the calculator press:

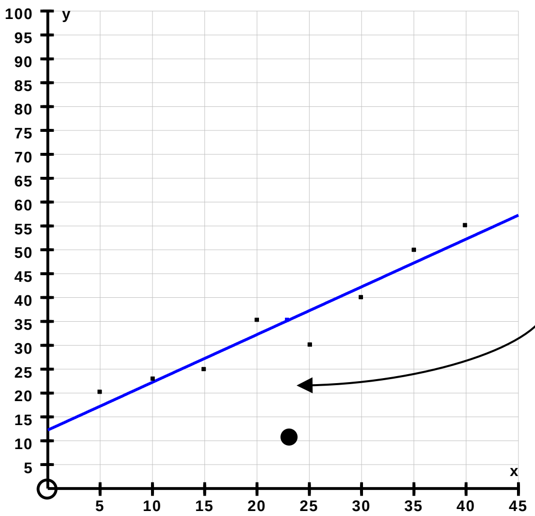
MODE 2
2: A+Bx
Enter the data points (column X and Y): Push = after each data point
Press AC.
SHIFT STAT
5: REG
1: A = (to determine the y- intercept of the line)
SHIFT STAT
5: REG
2: B = (to determine the gradient of the line)
SHIFT STAT
5: REG
3: r = (to determine correlation coefficient)

EXAMPLE

x	5	10	15	20	25	30	35	40
y	20	223	25	35	30	40	50	55

Using the calculator, the equation for the line of best fit (or regression line) can be determined giving:

$$y = 1x + 12{,}25$$



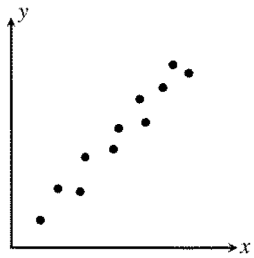
NB: The line of best fit ALWAYS goes through the point $(\bar{x}; \bar{y})$.

In this case it goes through the point (23 ; 35)

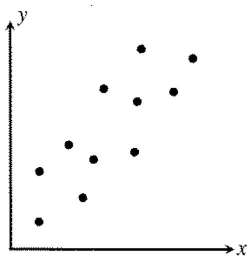
CORRELATION

The **strength of the relationship** between the two variables represented in a scatter diagram, depends on how close the points lie to the line of best fit. The closer the points lie to this line, the stronger the relationship or **correlation**.

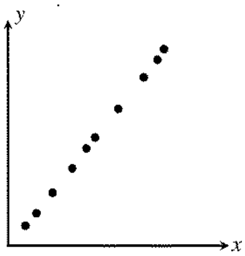
Correlation (tendency of the graph) can be described in terms of the general distribution of data points, as follows:



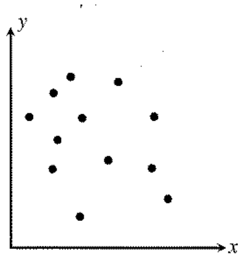
Strong positive Correlation



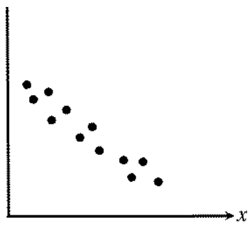
Fairly strong positive



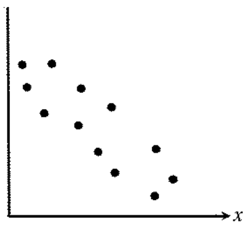
Perfect positive



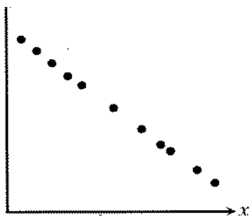
No correlation



Strong negative



Fairly strong negative



Perfect negative

CORRELATION COEFFICIENT

The correlation between two variables can also be described in terms of a number, called the correlation coefficient. The correlation coefficient, r, indicates the strength and the direction of the correlation between two variables. This number can be anything between -1 and 1.

r	Interpretation
$\mathbf{1}$	Perfect positive relationship
$\mathbf{0,9}$	Strong positive relationship
$\mathbf{0,5}$	Fairly strong positive relationship
$\mathbf{0,2}$	Weak positive relationship
$\mathbf{0}$	No relationship
$\mathbf{-0,2}$	Weak negative relationship
$\mathbf{-0,5}$	Fairly weak negative relationship
$\mathbf{-0,9}$	Strong negative relationship
$\mathbf{-1}$	Perfect negative relationship

Example

Refer to the previous example again.

For the given data set $r = 0{,}958$ which means that there is a strong positive relationship between the two variables.

Mixed Exercise on Statistics

1 A national soccer team has participated against teams of other countries in a competition for the past 14 years. Their results were as follows:

YEAR	MATCHES PLAYED	WINS	DRAWS	LOSSES	GOALS FOR	GOALS AGAINST
1999	5	3	2	0	11	3
2000	3	1	1	1	2	22
2001	5	3	1	1	10	4
2002	4	2	0	2	8	6
2003	7	2	3	2	5	4
2004	7	6	1	0	14	5
2005	5	2	0	3	8	7
2006	7	5	1	1	15	4
2007	6	1	2	3	9	11
2008	4	2	1	1	4	2
2009	3	1	1	1	2	3
2010	3	1	0	2	5	10
2011	1	0	0	1	2	3
2012	5	4	0	1	18	9

a Determine the quartiles for:

i the matches played

ii the wins

iii the goals scored against the soccer team.

b Draw a box and whisker plot for the goals against the soccer team and comment on the distribution of the data.

c Calculate the mean of the number of matches played.

d Calculated the standard deviation of the number of matches played.

2 Fifty people were asked what percentage of their December holiday expenses were related to transport costs. The responses were as follows:

PERCENTAGE	FREQUENCY (f)
$10 < x \leq 20$	6
$20 < x \leq 30$	14
$30 < x \leq 40$	16
$40 < x \leq 50$	11
$50 < x \leq 60$	3

a Draw an ogive to represent the data above.

b Use your ogive to determine the median percentage of the holiday expenses spent on travel expenses.

c Calculate the estimated mean.

d Calculate the standard deviation of the data.

3 An athlete's ability to take and use oxygen is called his VO_2 max. The following table shows the VO_2 max and the distance eleven atheletes can run in an hour.

VO_2 max	20	55	30	25	40	30	50	40	35	30	50
Distance(km)	8	18	13	10	11	12	16	14	13	9	15

a Represent the data on a scatter graph.

b Determine the equation of the line of best fit.

c Draw the line of best fit on the scatter graph.

d Use your line of best fit to predict the VO_2 max of an athlete that runs 19 km.

e Determine the correlation coefficient of the data and comment on the correlation.

4 Five number $4; 8; 10; x$ and y have a mean of 10 and a standard deviation of 4. Find x and y.

5 The standard deviation of five numbers is 7,5. Each number is increased by 2. What will the standard deviation of the new set of numbers be? Explain your answer.

Overview

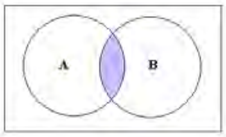
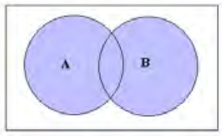
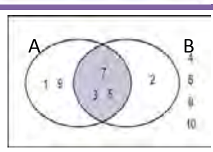
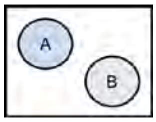
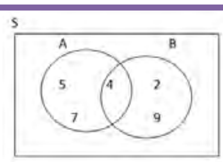
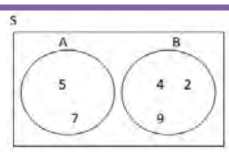
Chapter 12 Page 280 Probability	Unit 1 Page 282	
	Solving probability problems	• Venn diagrams • Tree diagrams • Two-way contingency tables
	Unit 2 Page 288	
	The counting principle	• The fundamental counting principle
	Unit 3 Page 292	
	The counting principle and probability	• Using the counting principle to calculate probability

REMEMBER YOUR STUDY APPROACH SHOULD BE:

1 Work through all examples in this chapter of your Learner's Book.
2 Work through the notes in this chapter of the study guide.
3 Do the exercises at the end of the chapter in the Learner's Book.
4 Do the mixed exercises at the end of this chapter in the study guide.

The only way to STUDY Maths is to DO Maths!

Probability

SUMMARY OF THEORY ON PROBABILITY		
CONCEPT/DEFINITION	MATHEMATICAL NOTATION/RULE	EXAMPLE
Probability = the chance that an event will occur	P	Values of probability can range from 0 to1 • For an event, K, that is certain NOT to happen $P(K) = 0$ • For an event, K that is CERTAIN to happen $P(K) = 1$
Sample Space = the set of all possible outcomes	S	
The number of elements in the sample space	$n(S)$	If $S = \{2; 4; 6\}$ then $n(S) = 3$
General rule for A and B inside the sample space S	$P(A or B) = P(A) + P(B) - P(A and B)$	
Intersection	$A\ and\ B$ or $A \cap B$	
Union	$A\ or\ B$ or $A \cup B$	
Inclusive events have elements in common	$P(A \cap B) \neq 0$	
Mutually exclusive/disjoint events DON'T INTERSECT, i.e. have NO elements in common	$P(A \cap B) = 0$ $\therefore P(A or B) = P(A) + P(B)$	
Exhaustive events = together they contain ALL elements of S	$\therefore P(A \cap B) = 1$	
Complement of A = all elements which are NOT in A	Complement of A = A'	
Complementary events = mutually exclusive and exhaustive (everything NOT in A, is in B)	$P(not\ A) = 1 - P(A)$ $P(A') = 1 - P(A)$ Or $P(A') + P(A) = 1$	
Independent events = outcome of 1st event DOES NOT influence the outcome of 2nd event	$P(A \cap B) = P(A) \times P(B)$	Tossing a coin and throwing a die
Dependent events = outcome of 1st event DOES influence the outcome of 2nd event	$P(A \cap B) \neq P(A) \times P(B)$	Choosing a ball from a bag, not replacing it, then choosing a 2nd ball

FACTORIAL NOTATION

The product $5 \times 4 \times 3 \times 2 \times 1$ can be written as $5!$

$$\therefore n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$$

The Fundamental Counting Principle	
RULE	EXAMPLE
RULE 1 **Where there are m ways to do one thing and n ways to do another, then there are $m \times n$ ways to do both**	a) You have 3 pants and 4 shirts. That means you have $3 \times 4 = 12$ different outfits.
RULE 2 **Where n different things have to be placed in n positions, the number of arrangements is $n!$**	b) 5 children have to be seated on 5 chairs in the front row of a class. The number of ways they can be seated is 5!=120
RULE 3 **Where n different things have to be placed in r positions, the number of arrangements is $\frac{n!}{(n-r)!}$**	c) 8 students participated in a 100 m race. The first three positions can be occupied in $\frac{8!}{(8-3)!} = \frac{8!}{5!} = 8 \times 7 \times 6 = 336$ ways.
RULE 4 **When seating b boys and g girls in a row, the number of arrangements are:** • **Boys and girls <u>in any order</u>: $(b+g)!$ ways** • **<u>Boys together</u> and <u>girls together</u>: $2 \times b! \times g!$ ways** • **<u>Only girls</u> together: $(b+1)! \times g!$ arrangements** • **<u>Only boys</u> together: $(g+1)! \times b!$ arrangements**	d) 3 girls and 4 boys have to be seated on 7 chairs, with girls together and boys together. Number of ways $= 2 \times 3! \times 4! = 288$ e)5 Maths books and 2 Science books have to be place on a shelf, but the Maths books have to be placed together. Number of ways $= (2+1)! \times 5! = 360$

LETTER ARRANGEMENTS

When making new words from the letters in a given word , one has to distinguish between:

Treating repeated letters as DIFFERENT letters.
The normal counting principle (Rule 2) applies here.

Treating repeated letters as IDENTICAL.

The following rule applies:

For n letters of which m_1 are identical, m_2 are identical, … and m_n are identical, the number of arrangements is given by:

$$\frac{n!}{m_1! \times m_2! \times \ldots \times m_n!}$$

Examples:

1 How many different arrangements can be made with the letters of the word MATHEMATICS, if repeated letters are treated as **different** letters.
The letters are regarded as 11 different letters.
Number of arrangements 11!

2 How many different arrangements can be made with the letters of the word MATHEMATICS, if repeated letters are treated as **identical**.
The letters are regarded as 11 different letters.
Number of arrangements $= \frac{11!}{2! \times 2! \times 2!} = 6\ 652\ 800$ (The M, A and T repeat)

Mixed Exercise on Probability

1. How many different 074- cell phone numbers are possible if the digits may not repeat?
2. How many different 082- cell phone numbers are possible if the digits may only be integers?
3. What is the probability that you will draw a queen of diamonds from a pack cards?
4. How many different arrangements can be made with the letters of the word TSITSIKAMMA, if:
   a. repeating letters are regarded as different letters
   b. repeating letters are regarded as identical.
5. Four different English books, three different German books and two different Afrikaans books are randomly arranged on a shelf.
   Calculate the number of arrangements if:
   a. the English books have to be kept together
   b. all books of the same language have to be kept together
   c. the order of the books does not matter.
6. In how many different ways can a chairman and a vice-chairman be chosen from a committee of 12 people?
7. The letters of the word MATHEMATICS have to be rearranged. Calculate the probability that the “word” formed will not start and end with the same letter.
8. In how many different ways can the letters of the word MATHEMATICS rearranged so that
   a. the H and the E stay together.
   b. the E keep its position.

Chapter 1: Number patterns, sequences and series

1 a $T_n = a + (n-1)d$

$a = 5\,;\ \ d = 4$

$T_n = 5 + (n-1)4 = 4n + 1$

b $217 = 4n + 1$

$4n = 216$

$\therefore n = 54$

2 a $9 = ar^4$

$729 = ar^8$

$\frac{729}{9} = \frac{ar^8}{ar^4}$

$r^4 = 81$

$r = \pm 3$

b $T_{10} = r \times T_9$

$T_{10} = \pm 2187$

3 a $T_2 - T_1 = T_3 - T_2$

$\big(5x - (2x-4)\big) = \big((7x-4) - 5x\big)$

$5x - 2x - 7x + 5x = -4 - 4$

$x = -8$

b $-20\,;\ -40\,;\ -60$

4 a $T_n = an^2 + bn + c$

$a = \frac{3}{2}$

$b = 5 - 3\left(\frac{3}{2}\right) = \frac{1}{2}$

$c = 2 - \frac{3}{2} - \frac{1}{2} = 0$

$T_n = \left(\frac{3}{2}\right)n^2 + \left(\frac{1}{2}\right)n$ Note: alternative methods can be used

b $260 = \left(\frac{3}{2}\right)n^2 + \left(\frac{1}{2}\right)n$

$3n^2 + n - 520 = 0$

$(3n + 40)(n - 13) = 0$

$n = 13$

$13th\ term\ is\ equal\ to\ 260.$

5 $T_n = a + (n-1)d$

$a = 17\ ;\ d = -3$

$-2785 = 17 + (n-1)(-3)$

$-2802 = (n-1)(-3)$

$934 = (n-1)$

$n = 935$

The sequence has 935 terms.

6 a $T_n = n^2$

b $T_n = an^2 + bn + c$

$a = 4 \div 2 = 2$

$b = 8 - 3(2) = 2$

$c = 4 - 2 - 2 = 0$

$\therefore\ T_n = 2n^2 + 2n$

7 a $T_1 = 3\ ;T_2 = -2\ ;T_3 = -7$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$a = 3\ ;d = -5$

$S_{30} = \frac{30}{2}[2(3) + (30-1)(-5)]$

$S_{30} = -2085$

b $T_1 = \frac{1}{2}\ ;T_2 = 1\ ;T_3 = 2$

$S_9 = \frac{\frac{1}{2}(2^9-1)}{2-1}$

$S_9 = 255{,}5$

8 $n = 6$

$T_n = 1 + (n-1)4 = 4n - 3$

$1 + 5 + 9 + \cdots + 21 = \sum_{k=1}^{6} 4k - 3$

9 a $T_5 = 0\ ;T_{13} = 12$

$0 = a + 4d$ ...(1)

$12 = a + 12d$ ...(2)

(2)-(1): $12 = 8d$

$d = \frac{3}{2}$

$a = -4\left(\frac{3}{2}\right) = -6$

b $S_{21} = \frac{21}{2}\left[2(-6) + (21-1)\left(\frac{3}{2}\right)\right]$

$S_{21} = 189$

10 a For it to be a converging sequence $-1 < r < 1$.

$r = \frac{T_2}{T_1} = \frac{(x^2-9)}{x+3}$

$r = \frac{(x+3)(x-3)}{x+3}$

$r = x - 3$

$\therefore\ -1 < x - 3 < 1$

$2 < x < 4$

b $S_\infty = \frac{a}{1-r}$

$13 = \frac{(x+3)}{1-(x-3)}$

$13 = \frac{(x+3)}{(-x+4)}$

$13(-x+4) = (x+3)$

$-13x + 52 = x + 3$

$-13x - x = 3 - 52$

$-14x = -49$

$x = \frac{7}{2}$

11 For series in numerator:

$99 = 1 + (n-1)2$

$n = 50$ terms

$S_{50} = \frac{50}{2}[2(1) + (50-1)2] = 2\ 500$

For series in denominator:

$299 = 201 + (n-1)2$

$n = 50$ terms

$S_{50} = \frac{50}{2}[2(201) + (50-1)2] = 12\ 500$

Value $= \frac{2\ 500}{12\ 500} = \frac{1}{5}$

12 $T_9 = S_9 - S_8$

$S_9 = 3(9)^2 - 2(9) = 225$

$S_8 = 3(8)^2 - 2(8) = 176$

$\therefore T_9 = 225 - 176 = 49$

13 a Let r =constant ratio

$7r^3 = 189$

$r^3 = 27$

$r = 3$

$x = 7 \times 3 = 21$

$y = 21 \times 3 = 63$

b $206\ 668 = \frac{7(3^n-1)}{3-1}$

$206\ 668 = \frac{7(3^n-1)}{2}$

$413\ 336 = 7(3^n - 1)$

$59\ 048 = 3^n - 1$

$3^n = 59\ 049$

$3^n = 3^{10}$

$\therefore n = 10$

Chapter 2: Functions

1 $2x - 3y = 17 \quad \ldots(1)$

$3x - y = 15 \quad \ldots(2)$

$(2) \times 3: 9x - 3y = 45 \quad \ldots(3)$

$(1) - (3): -7x = -28$

$x = 4$

Substitute into (1):

$2(4) - 3y = 17$

$y = -3$

Intercept is $(4; -3)$

2 a $y = mx + 3$

Substitute $(-3; 0)$

$0 = m(-3) + 3$

$m = 1$

$\therefore y = x + 3$

b $y = mx + 1$

Substitute $(2; -1)$:

$-1 = m(2) + 1$

$m = -1$

$g: y = -x + 1$

c $x + 3 = -x + 1$

$2x = -2$

$x = -1$

Substitute $x = -1$:

$y = -1 + 3 = 2$

$\therefore P(-1; 2)$

d Yes, because the products of their gradients is -1.

$(-1 \times 1 = -1)$

e $y = -x - 2$

3 a Let $y = 0$:

$0 = x^2 - 2x - 3$

$0 = (x - 3)(x + 1)$

$\therefore x = 3 \text{ or } x = -1$

$A(-1; 0)$ and $B(3; 0)$

Let $x = 0$:

$y = (0)^2 - 2(0) - 3$

$y = -3$

$\therefore C(0; -3)$

$OA = 1 \text{ unit}$

$OB = 3 \text{ units}$

$OC = 3 \text{ units}$

b $x = \frac{-b}{2a} = \frac{2}{2(1)} = 1$

Substitute $x = 1$:

$y = (1)^2 - 2(1) - 3 = -4$

$D(1; -4)$

c $c = -3$

$m = \frac{-3-0}{0-3}$

$m = 1$

d For the graph to have only one real root it has to move 4 units up.

$y = x^2 - 2x - 3 + 4 = x^2 - 2x + 1$

$\therefore k = 1$

4 a Let $y = 0$:

$0 = -2(x + 1)^2 + 8$

$0 = -2x^2 - 4x + 6$

$0 = (-2x + 2)(x + 3)$

$x = 1 \text{ or } x = -3$

$A(-3; 0)$ and $B(1; 0)$

$AB = 4$ units

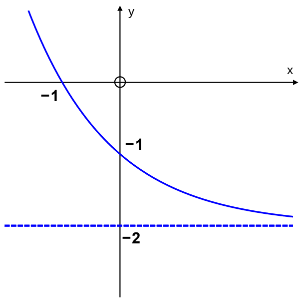
b $C(-1; 8)$

c $x = 0,\ y = 6$

$D(0; 6)\ \ E(-2; 6)$

$\therefore DE = 2$ units

5 a



b $x \in R$

c $x \leq -1$

6 a Substitute the point A into the equation $y = \frac{a}{x}$

$2 = \frac{a}{-2}$

$a = -4$

b $B(2; -2)$

c $y = \frac{-4}{x-1} + 2$

7 a $y = -(0)^2 - 2(0) + 8 = 8$

$A(0; 8)$

b $0 = -x^2 - 2x + 8$

$0 = (-x + 2)(x + 4)$

$x = 2 \;\; or \; x = -4$

$B(-4; 0)$ and $C(2; 0)$

c $D(-1; 0)$

$CD = 3 \; units$

d $x = -1$

$y = -(-1)^2 - 2(-1) + 8$

$y = -1 + 2 + 8 = 9$

$E(-1; 9)$

$DE = 9 \; units$

e $A(0; 8)$

$F(-2; 8)$

$AF = 2 \; units$

f $-x^2 - 2x + 8 = \frac{1}{2}x - 1$

$-2x^2 - 4x + 16 = x - 2$

$-2x^2 - 5x + 18 = 0$

$(2x + 9)(-x + 2) = 0$

$x = \frac{-9}{2} \; or \; x = 2$

$x = \frac{-9}{2}$ at H

Substitute $x = \frac{-9}{2}$ into the equation $y = \frac{1}{2}x - 1$

$y = \frac{1}{2}\left(\frac{-9}{2}\right) - 1$

$y = -\frac{13}{4}$

$\therefore G\left(\frac{-9}{2}; \frac{-13}{4}\right)$

$GH = 3{,}25\ units$

g $\quad f(x) - g(x) = -x^2 - 2x + 8 - \left[\frac{1}{2}x - 1\right]$

$= -x^2 - \frac{5}{2}x + 9$

Minimum at turning point:

$x = \frac{\frac{5}{2}}{-2} = -\frac{5}{4}$

h $\quad RS_{max} = -\left(-\frac{5}{4}\right)^2 - \frac{5}{2}\left(\frac{-5}{4}\right) + 9 = \frac{169}{16}$

i $\quad f(x) - g(x) > 0 \qquad \therefore f(x) > g(x)$

$-\frac{9}{2} < x < 2$

8 a $\quad y = -4$

b $\quad y = b^x + c$

$c = -4$

$y = b^x - 4$

Substitute the point $(2; 5)$ into the equation:

$5 = b^2 - 4$

$b^2 = 9$

$b = 3$

$y = 3^x - 4$

c $\quad y = -1\ ;\ x = -2$

d $\quad y = \frac{a}{x+2} - 1$

Substitute the point $A(0; -3)$:

$-3 = \frac{a}{0+2} - 1$

$-3 = \frac{a}{2} - 1$

$\frac{a}{2} = -2$

$a = -4$

$y = \frac{-4}{x+2} - 1$

e $\quad$ Substitute $(-2; -1)$ into $y = x + k_1$ and $y = -x + k_2$

$-1 = -2 + k_1 \qquad -1 = 2 + k_2$

$k_1 = 1 \qquad k_2 = -3$

$y = x + 1 \qquad y = -x - 3$

f $\quad x > -2; x \neq 0$

9 a $y = 2x^2$

$x = 2y^2$

$y^2 = \frac{x}{2}$

$y = \pm\sqrt{\frac{x}{2}}$

b $x \leq 0 \; or \; x \geq 0$

10 a $y = a^x$

Substitute point A:

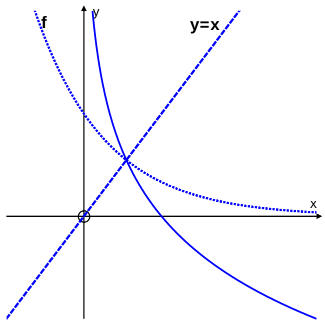
$3 = a^{-1}$

$a = \frac{1}{3}$

$y = \left(\frac{1}{3}\right)^x$

b $f^{-1}: y = \log_{\left(\frac{1}{3}\right)} x$

c



d $x > 0$

11 x –intercept: $(3; 0)$

y –intercept: $(0; -2)$

Chapter 3: Logarithms

1 a $x = 3^2 = 9$

b $x = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$

c $\log_4 x = -2$

$x = (4)^{-2} = \frac{1}{(4)^2} = \frac{1}{16}$

d $x = (5)^{-2} = \frac{1}{(5)^2} = \frac{1}{25}$

e $x^3 = 10^6$

$x = 10^2$

$x = 100$

f $81 = 3^x$

$3^x = 3^4$

$x = 4$

g $\frac{1}{9} = 3^x$

$3^x = 3^{-2}$

$x = -2$

2 a Substitute $\left(2;\frac{9}{4}\right)$: $\frac{9}{4} = a^2$

$a = \frac{3}{2}$

b $f^{-1}: y = \log_{\left(\frac{3}{2}\right)} x$

c $g(x) = \left(\frac{3}{2}\right)^{-x}$

d $h(x) = -\log_{\left(\frac{3}{2}\right)} x$

3 a i) $g(x) = -\log_2 x$

ii) $p(x) = \log_2(-x)$

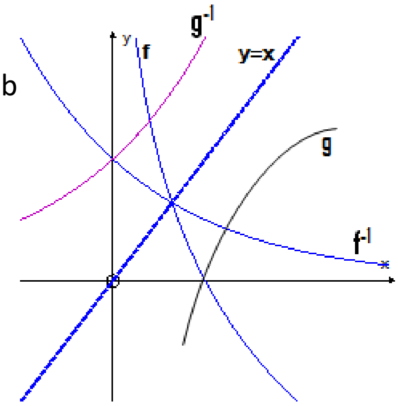
iii) $q(x) = -\log_2(-x)$

iv) $f^{-1}: y = 2^x$

v) $g^{-1}: y = 2^{-x}$

vi) $h(x) = \log_2(x + 2)$

b



c For f^{-1} and g^{-1}:

Domain $x \in R$;

Range $y > 0$

4 a y –coordinate $= 0$

$0 = \log_b x$

$x = b^0 = 1$

$A(1; 0)$

b Because graph is increasing as x increases.

c Substitute B: $\frac{3}{2} = \log_b 8$

$8 = b^{\frac{3}{2}}$

$(8)^{\frac{2}{3}} = \left(b^{\frac{3}{2}}\right)^{\frac{2}{3}}$

$b = (8)^{\frac{2}{3}} = (2^3)^{\frac{2}{3}} = 2^2 = 4$

d $g(x) = 4^x$

e Substitute $y = -2$: $-2 = \log_4 x$

$x = 4^{-2} = \frac{1}{16}$

Chapter 4: Finance, growth and decay

1 a $A = P(1 + i.n)$

$A = 15\ 000(1 + (0{,}106)(5))$

$A = R22\ 950$

b $A = P(1 + i)^n$

$A = 15\ 000(1 + (0{,}024))^{20}$

$A = R24\ 104{,}07$

It is better to invest it at 9.6% p.a , interest compounded quarterly.

2 a Nominal interest rate

b $A = P(1 + i)^n$

$95\ 000 = P\left(1 + \frac{0{,}085}{12}\right)^{60}$

$P = \frac{95\ 000}{\left(1+\frac{0{,}085}{12}\right)^{60}}$

$P = R62\ 202{,}48$

c $R95\ 000 - R62\ 202{,}48 = R32\ 797{,}52$

3 a $A = P(1 + i)^n$

$A = 8\ 000(1 + 0.06)^2$

$A = R8\ 988{,}80$

b $F = \frac{x[(1+i)^n - 1]}{i}[1 + i]$

$F = \frac{2\ 000\left[\left(\frac{0{,}07}{2}\right)^4 - 1\right]}{0.035}\left[1 + \frac{0{,}07}{2}\right]$

$F = R8\ 724{,}93$

She will NOT have enough money to buy the TV in two years.

4 a $1 + i_{eff} = \left(1 + \frac{i_{nom}}{m}\right)^m$

$1 + i_{eff} = \left(1 + \frac{0.0785}{12}\right)^{12}$

$i_{eff} = 0.08138 \ldots$

$Eff.rate = 8.14\%$

b $1+i_{ef}=\left(1+\frac{i_{nom}}{m}\right)^m$

$1+0{,}0925=\left(1+\frac{i_{nom}}{4}\right)^4$

$\sqrt[4]{1{,}0925}=\left(1+\frac{i_{nom}}{4}\right)$

$1.022-1=\frac{i_{nom}}{4}$

$i_{nom}=0.0894\ldots$

Nom. rate= $8{,}95\%$ p.a. compounded quarterly

5 $A=P(1+i)^n$

$179\,200=350\,000(1-i)^3$

$0{,}512=(1-i)^3$

$1-i=\sqrt[3]{0{,}512}$

$i=0.2$

Dep. rate= 20%

6 $A=\left[20\,000\left(1+\frac{0.0975}{4}\right)^7+10\,000\left(1+\frac{0.0975}{4}\right)\right]\left(1+\frac{0.0995}{12}\right)^{15}$

$A=[23\,672.43+10\,243.75](1.13185\ldots)$

$A=R38\,388{,}36$

OR

$A=\left[20\,000\left(1+\frac{0.0975}{4}\right)^6+10\,000\right]\left(1+\frac{0.0975}{4}\right)\left(1+\frac{0.0995}{12}\right)^{15}$

$A=[23\,109.142+10\,000](1.024375)(1.13\ldots)$

$A=R38\,388{,}36$

7 a $A=900\,000(1-0.15)^5$

$A=R399\,334{,}78$

$A=900\,000(1+0.18)^5$

$A=R2\,058\,981{,}98$

$R2\,058\,981{,}98-R399\,334{,}78=R1\,659\,647{,}20$

b $1\,659\,647{,}20=\frac{x[(1+0.02)^{61}-1]}{0{,}02}$

$x=\frac{0{,}02\times 1659647{,}20}{[(1+0{,}02)^{61}-1]}$

$x=R14\,144{,}81$

8 $A=P(1+i.n)$ 2 yrs = 24 months

$(24\times 85)=1\,500(1+i.2)$

$i=0.18$

rate= 18%

9 a $P = \frac{x[1-(1+i)^{-n}]}{i}$

$P = \frac{(6\ 500)[1-(1+0{,}01)^{-240}]}{0{,}01}$

$P = R590\ 326{,}21$

b $P = \frac{(6\ 500)[1-(1+0{,}01)^{-96}]}{0{,}01}$

$P = R399\ 930{,}07$

10 $F = \frac{1\ 000\left[\left(1+\frac{0{,}01}{12}\right)^{73}-1\right]}{\frac{0{,}01}{12}}$

$F = R99\ 915{,}81$

$A = P(1+i)^n$

$A = 99\ 915{,}81\left(1+\frac{0{,}01}{12}\right)^5$

$A = R104\ 147{,}21$

11 $F = \frac{x[(1+i)^n-1]}{i}$

$48\ 000 = \frac{300\left[\left(1+\frac{0{,}09}{12}\right)^n-1\right]}{\frac{0{,}09}{12}}$

$2{,}2 = (1{,}0075)^n$

$n = \log_{1{,}0075} 2{,}2$

$n = 106$

8 years and 10 months

12 $A = P(1+i.n)$

$A = 13\ 500\big(1+(0.12)(4)\big)$

$A = R19\ 980$

Repayment = $R19\ 980 \div 48 = R416{,}25$

Including insurance= $R416{,}25 + R30 = R446{,}25$

13 $A = 400\ 000(1+0{,}02)^4$

$A = R432\ 972{,}86$ (amount owing after 1 year)

$P = \frac{x[1-(1+i)^{-n}]}{i}$

$432\ 972{,}86 = \frac{x[1-(1+0{,}02)^{-16}]}{0{,}02}$

$x = R31\ 888{,}51$

Chapter 5: Compound angles

1 a $2\cos 2x = -1$

$\therefore cos2x = -\frac{1}{2}$

$2x = \pm 120° + k.360°; k \in Z$

$\therefore x = \pm 60° + k.180°; k \in Z$

b $sinx = 3cosx$

$\frac{sinx}{cosx} = 3$

$tanx = 3$

$\therefore x = 71{,}47° + k.180°; k \in Z$

c $sinx = cos3x$

$\therefore \cos(90° - x) = cos3x$

$90° - x = \pm 3x + k.360°$

$4x = 90° + k.360°$ or $2x = -90° + k.360°$

$x = 22{,}5° + k.90°$ $x = -45° + k.180°; k \in Z$

d $6 - 10cosx - 3(1 - cos^3x) = 0$

$\therefore 3cos^3x - 10cosx + 3 = 0$

$\therefore (3cosx - 1)(cosx - 3) = 0$

$\therefore cosx = \frac{1}{3}$ or $cosx = 3$ (no solution)

$\therefore x = \pm 70{,}53° + k.360°; k\epsilon Z$

For $x \in [-360°; 360°]$ $x \in \{-289{,}47°; -70{,}53°; 289{,}47°\}$

e $2(sin^2x + cos^2x) - sinxcosx - 3cos^2x = 0$

$2sin^2x - sinxcosx - cos^2x = 0$

$(2sinx + cosx)(sinx - cosx) = 0$

$tanx = -\frac{1}{2}$ or $tanx = 1$

$x = -26{,}57° + k.180°$ or $x = 45° + k.180°; k \in Z$

f $3(sin^2x + cos^2x) - 8sinx + 16sinxcosx - 6cosx = 0$

$\therefore 3 - 6cosx - 8sinx + 16sinxcosx = 0$

$3(1 - 2cosx) - 8sinx(1 - 2xosx) = 0$

$(1 - 2cosx)(3 - 8sinx) = 0$

$cosx = \frac{1}{2}$ or $sinx = \frac{3}{8}$

$\therefore x = \pm 60° + k.360°$ or $x = 22{,}02° + k.360°$ or $x = 157{,}98° + k.360°; k \in Z$

2 a LHS= $\cos x + \dfrac{\sin x}{\cos x} \times \sin x$

$$= \frac{\cos^2 x + \sin^2 x}{\cos x}$$

$$= \frac{1}{\cos x}$$

=RHS

Not valid for $x = 90^0 + k.180^0;\ k \in Z$

b LHS = $\dfrac{\sin^2\theta - \cos\theta(1-\cos\theta)}{(1-\cos\theta)\sin\theta} = \dfrac{\sin^2\theta + \cos^2\theta - \cos\theta}{(1-\cos\theta)\sin\theta} = \dfrac{1-\cos\theta}{(1-\cos\theta)\sin\theta} = \dfrac{1}{\sin\theta}$

=RHS

Not valid for $\theta = k.180^0;\ k \in Z$

c LHS = $\dfrac{1-\cos^2 x}{\cos x} = \dfrac{\sin^2 x}{\cos x} = \dfrac{\sin x}{\cos x}.\sin x = \tan x.\sin x$ = RHS

Not valid for $x = 90^0 + k.180^0;\ k \in Z$

d LHS = $\dfrac{\sin x(\sin^2 x + \cos^2 x)}{\cos x} = \dfrac{\sin x}{\cos x} = \tan x$ =RHS

Not valid for $x = 90^0 + k.180^0;\ k \in Z$

e

$$LHS = \frac{1+\dfrac{\sin x}{\cos x}}{1-\dfrac{\sin x}{\cos x}} = \frac{\cos x + \sin x}{\cos x} \times \frac{\cos x}{\cos x - \sin x} \times \frac{\cos x + \sin x}{\cos x + \sin x}$$

$$= \frac{\cos^2 x + 2\sin x\cos x + \sin^2 x}{\cos^2 x - \sin^2 x} = \frac{1+2\sin x\cos x}{\cos^2 x - \sin^2 x} = RHS$$

Not valid for $x = \pm 45^0 + k.180^0;\ k \in Z$

f $\quad LHS = \sin(45° + x).\sin(45° - x)$

$= (\sin 45°.\cos x + \sin x.cos45°) \times (\sin 45°.\cos x - \sin x.\cos 45°)$

$= \left(\frac{\sqrt{2}}{2}\cos x + \frac{\sqrt{2}}{2}\sin x\right)\left(\frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x\right)$

$= \left(\frac{\sqrt{2}}{2}\cos x\right)^2 - \left(\frac{\sqrt{2}}{2}\sin x\right)^2$

$= \frac{1}{2}cos^2x - \frac{1}{2}sin^2x$

$= \frac{1}{2}(cos^2x - sin^2x)$

$= \frac{1}{2}\cos 2x$

$= RHS$

g $\quad LHS = \frac{\sin 2\theta - \cos\theta}{\sin\theta - \cos 2\theta}$

$= \frac{2\sin\theta.\cos\theta - \cos\theta}{\sin\theta - (1 - 2sin^2\theta)}$

$= \frac{\cos\theta(2\sin\theta - 1)}{2sin^2\theta + \sin\theta - 1}$

$= \frac{\cos\theta(2\sin\theta - 1)}{(2\sin\theta - 1)(\sin\theta + 1)}$

$= \frac{\cos\theta}{\sin\theta + 1}$

$= RHS$

h $\quad LHS = \frac{\cos x - \cos 2x + 2}{3\sin x - \sin 2x}$

$= \frac{\cos x - (2cos^2x - 1) + 2}{3\sin x - 2\sin x.\cos x}$

$= \frac{-2cos^2x + \cos x + 3}{3\sin x - 2\sin x.\cos x}$

$= \frac{(-2\cos x + 3)(\cos x + 1)}{\sin x(3 - 2\cos x)}$

$= \frac{\cos x + 1}{\sin x}$

$= RHS$

3 a $$\frac{\sin(180^0 - x)\tan(-x)}{\tan(180^0 + x)\cos(x - 90^0)} = \frac{\sin x(-\tan x)}{\tan x(\sin x)} = -1$$

b $$\frac{\sin(180^0 + x)\tan(x - 360^0)}{\tan(360^0 - x)(-\cos 60^0)(\tan 45^0)} = \frac{\sin x.\tan x}{-\tan x(-0,5)(1)} = 2\sin x$$

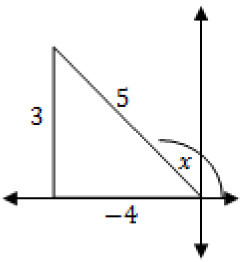
4 a $\cos 73^0 = \cos(90^0 - 17^0) = \sin 17^0 = k$

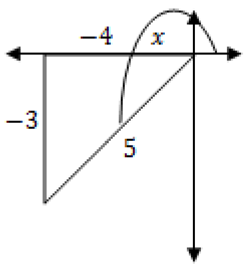
b $\cos(-163°) = cos163° = -cos17° = -\sqrt{1-k^2}$

c $tan197° = tan17° = \frac{k}{\sqrt{1-k^2}}$

d $cos326° = cos34° = cos2(17°) = 1 - 2sin^2 17° = 1 - 2k^2$

5 a $cosx = -\frac{4}{5}$





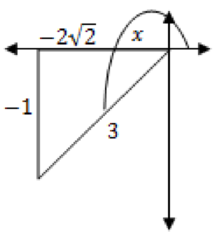
$$5\sin x + 3\tan x = 5\left(\frac{3}{5}\right) + 3\left(\frac{3}{-4}\right) \quad \text{or} \quad = 5\left(\frac{-3}{5}\right) + 3\left(\frac{-3}{-4}\right)$$

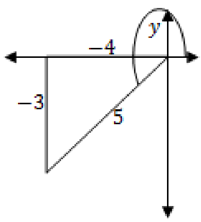
$$= 3 - \frac{9}{4} = \frac{3}{4} \quad \text{or} \quad = -5 + \frac{9}{4} = -\frac{3}{4}$$

b $$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

$$\therefore \tan 2x = \frac{2\left(\frac{3}{-4}\right)}{1 - \left(\frac{3}{-4}\right)^2} = -\frac{3}{2} \times \frac{16}{6} = -\frac{24}{7} \quad \text{or} \quad \tan 2x = \frac{3}{2} \times \frac{16}{7} = \frac{24}{7}$$

6 a





$\cos(x-y) = \cos x.\cos y + \sin x.\sin y$

$= \frac{-2\sqrt{2}}{3} \times \frac{-4}{5} + \frac{-1}{3} \times \frac{-3}{5}$

$= \frac{8\sqrt{2}}{15} + \frac{1}{5}$

$= \frac{8\sqrt{2}+3}{15}$

b $\cos 2x - \cos 2y$

$= 1 - 2sin^2x - (1 - 2sin^2y)$

$= 2sin^2y - 2sin^2x$

$= 2\left(\frac{-3}{5}\right)^2 - 2\left(\frac{-1}{3}\right)^2$

$= \frac{18}{25} - \frac{2}{9} = \frac{112}{225}$

7 a $\cos 2(22{,}5°) = \cos 45° = \frac{1}{\sqrt{2}}$

b $\frac{1}{2} \times 2\sin 22{,}5°.\cos 22{,}5° = \frac{1}{2} \times \sin 2(22{,}5°)$

$= \frac{1}{2}\sin 45° = \frac{\sqrt{2}}{4}$

c $\sin 2(15°) = \sin 30° = \frac{1}{2}$

Chapter 6: Solving problems in three dimensions

1 a $In\ \Delta ABE : \tan\alpha = \frac{2h}{BE}$ $\quad\therefore BE = \frac{2h}{\tan\alpha}$

b $In\ \Delta CED$: $\tan(90° - \alpha) = \frac{h}{DE}$ $\quad\therefore DE = h\tan\alpha$

$In\ \Delta BDE$:

$BD^2 = BE^2 + ED^2 - 2(BE)(ED).\cos E$

$= (2h.\cot\alpha)^2 + (h.\tan\alpha)^2 - 2(2h.\cot\alpha)(h.\tan\alpha)\cos 120°$

$= 4h^2.cot^2\alpha + h^2tan^2\alpha - 4h^2(\cot\alpha\,.\tan\alpha)\left(-\frac{1}{2}\right)$

$= h^2(4cot^2\alpha + tan^2\alpha + 2)$

$= h^2\left(\frac{4}{tan^2\alpha} + tan^2\alpha + 2\right)$

$= \frac{h^2(tan^4\alpha+2tan^2\alpha+4)}{tan^2\alpha}$

$BD = \frac{h\sqrt{tan^4\alpha+2tan^2\alpha+4}}{\tan\alpha}$

c $h = \frac{BD\tan\alpha}{\sqrt{tan^4\alpha+2tan^2\alpha+4}}$

$= \frac{509.\tan 42°}{\sqrt{tan^4 42°+2tan^2 42°+4}}$

$CD = 182{,}90\ m$

2 a $C\hat{D}B = 180° - \theta - 30° = 150° - \theta$

b $\tan\theta = \frac{p}{CB}$ $\therefore p = CB.\tan\theta$

$\frac{CB}{\sin(150°-\theta)} = \frac{8}{\sin\theta}$

$CB = \frac{8.\sin(150°-\theta)}{\sin\theta}$

$= \frac{8.\sin(180°-(150°-\theta))}{\sin\theta}$

$= \frac{8.\sin(30°+\theta)}{\sin\theta}$

$p = \left(\frac{8.\sin(30°+\theta)}{\sin\theta}\right)\tan\theta = \frac{8.\sin(30°+\theta)}{\cos\theta}$

3 $AD = 13(Pythagoras)$

$\hat{A} = 180° - (\alpha + \beta)$

$\therefore \frac{CD}{\sin[180°-(\alpha+\beta)]} = \frac{13}{\sin\alpha}$

$\therefore \frac{CD}{\sin(\alpha+\beta)} = \frac{13}{\sin\alpha}$

$\therefore CD = \frac{13\sin(\alpha+\beta)}{\sin\alpha}$

4 a $Area\ \Delta ADC = \frac{1}{2} m.p \sin(180° - \theta)$

b $Area\ \Delta BDC = \frac{1}{2} n.p \sin\theta$

$Area\ \Delta ABC = Area\ \Delta ADC + Area\ \Delta BDC$

$= \frac{1}{2} mp \sin(180° - \theta) + \frac{1}{2} np \sin\theta$

$= \frac{1}{2} mp . \sin\theta + \frac{1}{2} np . \sin\theta$

$= \frac{1}{2} p(m + n) \sin\theta$

c $12{,}6 = \frac{1}{2}(8{,}1)(5{,}9) \sin\theta$

$\sin\theta = 0{,}527306968 \ldots$

$\theta = 31{,}82° \quad OR \quad \theta = 180° - 31{,}82° = 148{,}18°$

5 a $\sin\theta = \frac{p}{BC}$

$\therefore BC = \frac{p}{\sin\theta}$

b $\hat{B}_1 = 180° - 2\alpha$

c $\frac{AC}{\sin\hat{B}_1} = \frac{BC}{\sin A}$

$\frac{AC}{\sin(180° - 2\alpha)} = \frac{\frac{p}{\sin\theta}}{\sin\alpha}$

$AC = \frac{p\sin(180° - 2\alpha)}{\sin\theta . \sin\alpha} = \frac{p . \sin 2\alpha}{\sin\theta . \sin\alpha}$

6 a $\hat{R} = 180° - 30° - (150° - \alpha) = \alpha$

$\frac{12}{\sin\hat{R}} = \frac{QR}{\sin(150° - \alpha)}$

$\frac{12}{\sin\alpha} = \frac{QR}{\sin(30° + \alpha)}$

$QR = \frac{12\sin(30° + \alpha)}{\sin\alpha}$

$= \frac{12(\sin 30°.\cos\alpha + \cos 30°.\sin\alpha)}{\sin\alpha}$

$= \frac{12\left(\frac{1}{2}.\cos\alpha + \frac{\sqrt{3}}{2}\sin\alpha\right)}{\sin\alpha}$

$= \frac{6(\cos\alpha + \sqrt{3}\sin\alpha)}{\sin\alpha}$

b $\hat{P} = 180° - 90° - \alpha = 90° - \alpha$

$\frac{QR}{\sin\hat{P}} = \frac{PQ}{\sin\alpha}$

$\frac{\frac{6(\cos\alpha + \sqrt{3}\sin\alpha)}{\sin\alpha}}{\sin(90° - \alpha)} = \frac{PQ}{\sin\alpha}$

$PQ\sin(90° - \alpha) = \frac{\sin\alpha.6(\cos\alpha + \sqrt{3}\sin\alpha)}{\sin\alpha}$

$PQ = \frac{6\cos\alpha}{\cos\alpha} + \frac{6\sqrt{3}\sin\alpha}{\cos\alpha}$

$PQ = 6 + 6\sqrt{3}\tan\alpha$

c $23 = 6 + 6\sqrt{3}\tan\alpha$

$17 = 6\sqrt{3}\tan\alpha$

$tan\alpha = 1{,}64$

$\alpha = 58{,}56°$

Chapter 7: Polynomials

1 a $27x^3 - 8 = (3x - 2)(9x^2 + 6x + 4)$

b $5x^3 + 40 = 5(x^3 + 8) = 5(x + 2)(x^2 - 2x + 4)$

c $x^3 + 3x^2 + 2x + 6$

$= x^2(x + 3) + 2(x + 3)$

$= (x + 3)(x^2 + 2)$

d $4x^3 - x^2 - 16x + 4$

$= x^2(4x - 1) - 4(4x - 1)$

$= (4x - 1)(x^2 - 4)$

$= (4x - 1)(x - 2)(x + 2)$

e $4x^3 - 2x^2 + 10x - 5$

$= 2x^2(2x - 1) + 5(2x - 1)$

$= (2x - 1)(2x^2 + 5)$

f $x^3 + 2x^2 + 2x + 1$

$= (x^3 + 1) + (2x^2 + 2x)$

$= (x + 1)(x^2 - x + 1) + 2x(x + 1)$

$= (x + 1)(x^2 + x + 1)$

g $x^3 - x^2 - 22x + 40$

$= (x - 2)(x^2 + x - 20)$

$= (x - 2)(x + 5)(x - 4)$

h $x^3 + 2x^2 - 5x - 6$

$= (x - 2)(x^2 + 4x + 3)$

$= (x - 2)(x + 3)(x + 1)$

i $3x^3 - 7x^2 + 4$

$= (x - 1)(3x^2 - 4x - 4)$

$= (x - 1)(3x + 2)(x - 2)$

j $x^3 - 19x + 30$

$= (x - 2)(x^2 + 2x - 15)$

$= (x - 2)(x + 5)(x - 3)$

k $x^3 - x^2 - x - 2$

$= (x - 2)(x^2 + x + 1)$

2 a $x(x^2 + 2x - 4) = 0$

$x = 0 \quad or \quad x = -1 \pm \sqrt{5}$

b $(x - 2)(x^2 - x - 3) = 0$

$x = 2 \quad or \quad x = \frac{1 \pm \sqrt{3}}{2}$

c $(2x^3 - 12x^2) - (x - 6) = 0$

$2x^2(x - 6) - (x - 6) = 0$

$(x - 6)(2x^2 - 1) = 0$

$x = 6 \quad or \quad x = \pm\sqrt{\frac{1}{2}}$

d $(2x^3 - x^2) - (8x - 4) = 0$

$x^2(2x - 1) - 4(2x - 1) = 0$

$(x^2 - 4)(2x - 1) = 0$

$x = 2 \quad or\ x = -2 \quad or\ x = \frac{1}{2}$

e $(x - 1)(x^2 + 2x + 2) = 0$

$x = 1$

f $(x + 2)(x^2 - 2x - 8) = 0$

$(x + 2)(x - 4)(x + 2) = 0$

$x = -2 \quad or \quad x = 4$

g $(x^3 - 20x) + (3x^2 - 60) = 0$

$x(x^2 - 20) + 3(x^2 - 20) = 0$

$(x^2 - 20)(x + 3) = 0$

$x = \pm 2\sqrt{5} \quad or\ x = -3$

3 $f(3) = 3^3 - 3^2 - 5(3) - 3$

$= 27 - 9 - 15 - 3 = 0$

$(x - 3)$ is a factor

$(x - 3)(x^2 + 2x + 1) = 0$

$(x - 3)(x + 1)^2 = 0$

$x = 3 \quad or\ x = -1$

4 $g\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 8\left(\frac{1}{2}\right)^2 - \frac{1}{2} + 2$

$= \frac{1}{2} - 2 - \frac{1}{2} + 2 = 0$

$(2x - 1)(2x^2 - 3x - 2) = 0$

$(2x - 1)(2x + 1)(x - 2) = 0$

$x = \frac{1}{2} \quad or \quad x = -\frac{1}{2} \quad or\ x = 2$

Chapter 8: Differential calculus

1 a $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \frac{1-(x+h)^2-(1-x^2)}{h}$

$= \lim_{h\to 0} \frac{1-(x^2+2xh+h^2)-(1-x^2)}{h}$

$= \lim_{h\to 0} \frac{-2xh-h^2}{h}$

$= \lim_{h\to 0} \frac{h(-2x-h)}{h}$

$= \lim_{h\to 0} (-2x - h)$

$= -2x$

b $f'(x) = \lim_{h\to0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to0}\frac{-3(x+h)^2-(-3x^2)}{h}$

$= \lim_{h\to0}\frac{-3(x^2+2xh+h^2)-(-3x^2)}{h}$

$= \lim_{h\to0}\frac{-6xh-3h^2}{h}$

$= \lim_{h\to0}\frac{h(-6x-3h)}{h}$

$= \lim\limits_{h\to0}(-6x-3h)$

$= -6x$

2 a $y = \sqrt{x} - \frac{1}{2x^2} = x^{\frac{1}{2}} - \frac{1}{2}x^{-2}$

$\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} - \frac{1}{2} \times -2x^{-3}$

$= \frac{1}{2}x^{-\frac{1}{2}} + x^{-3}$

$= \frac{1}{2\sqrt{x}} + \frac{1}{x^3}$

b $D_x\left[\frac{2x^2-x-15}{x-3}\right]$

$= D_x\left[\frac{(2x+5)(x-3)}{x-3}\right]$

$= D_x[2x+5] = 2$

3 $f(x) = -2x^3 + 3x^2 + 32x + 15$

$f(-2) = -2(-2)^3 + 3(-2)^2 + 32(-2) + 15 = -21$

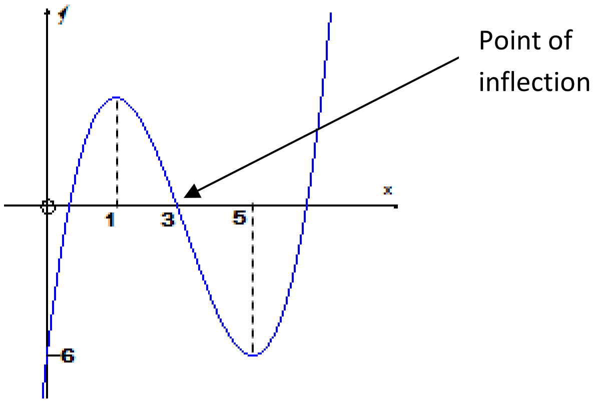
$f'(x) = -6x^2 + 6x + 32$

$f'(-2) = -6(-2)^2 + 6(-2) + 32 = -4$

Sub $(-2; -21)$ into $y = -4x + c$

$-21 = -4(-2) + c$ $\quad c = -29$ $\quad y = -4x - 29$

4



5 (2; 9) is a point on the graph and a turning point

$\therefore f(2) = 9$ and $f'(2) = 0$

$f'(x) = 3ax^2 + 10x + 4$

$0 = 3a(2)^2 + 10(2) + 4$

$\therefore a = -2$

$9 = (-2)(2)^3 + 5(2)^2 + 4(2) + b$

$\therefore b = -3$

6 a Turning point where $f'(x) = 0$ $\quad \therefore x = -2$ and $x = 5$

b Point of inflections is where $f''(x) = 0$, therefor where graph of f' turns

c f will decrease where its gradient f' is negative ($f' < 0$)

$-2 < x < 5$

7 a The graph bounces at $x = 1$ and has an x-intercept at $x = -1$

$\therefore f(x) = (x + 1)(x - 1)^2$

$f(x) = (x + 1)(x^2 - 2x + 1) = x^3 - x^2 - x + 1$

$\therefore a = -1; b = -1; c = 1$

b B is a turning point where $f'(x) = 0$

$3x^2 - 2x - 1 = 0$

$(3x + 1)(x - 1) = 0)$

At B $x = -\frac{1}{3}$

$f\left(-\frac{1}{3}\right) = \left(-\frac{1}{3}\right)^3 - \left(-\frac{1}{3}\right)^2 - \left(-\frac{1}{3}\right) + 1 = \frac{32}{27}$

$B\left(-\frac{1}{3}; \frac{32}{27}\right)$

8 a If $s(t)$ is distance, then $s'(t)$ is speed.

$s'(t) = 3t^2 - 4t + 3$

b Speed is a minimum where $s''(t) = 6t - 4 = 0$

$t = \frac{2}{3}$

c $6t - 4 = 8$

$t = 2s$

9 a Volume $= 2x^2h = 24$

$h = \frac{12}{x^2} = 12x^{-2}$

b $C(x) = 2x^2 \times 25 + 2x^2 \times 20 + 2 \times xh \times 20 + 2 \times 2xh \times 20$

$= 90x^2 + 120xh$

$= 90x^2 + 120x(12x^{-2})$

$= 90x^2 + 1440x^{-1}$

c $C'(x) = 180x - 1440x^{-2} = 0$

$180x - \frac{1440}{x^2} = 0$

$180x^3 - 1440 = 0$

$x^3 = 8$

$x = 2$

Chapter 9: AnalyticalgGeometry

1 a $m_{AB} = m_{CD}$

$\frac{1-(-4)}{-2-p} = \frac{0-2}{5-3}$

$\frac{5}{-2-p} = \frac{-2}{2} = -1$

$2 + p = 5$

$p = 3$

b $AB = \sqrt{(3-(-2))^2 + (-4-1)^2} = 5\sqrt{2}$

$CD = \sqrt{(5-3)^2 + (0-2)^2} = 2\sqrt{2}$

$AB: CD = 5\sqrt{2}: 2\sqrt{2} = 5: 2$

c $m_{NB} = m_{CD}$

$\frac{y+4}{x-3} = -1 \qquad \therefore y = -x - 1 \quad ...(1)$

$m_{ND} = m_{BC}$

$\frac{y-2}{x-3} = 2 \qquad \therefore y = 2x - 4 \quad ...(2)$

(1)-(2): $0 = 3x - 3$

$x = 1 \qquad y = -2$

$N(1; -2)$

d B and D have the same x-coordinate, so it is a vertical line with equation $x = 3$.

e Angle of inclination of a vertical line is 90°.

f Area of parallelogram = base x perpendicular height

$Area\ of\ NBCD = CD \times perp\ height$

$= 2\sqrt{2} \times 6 = 12\sqrt{2}$

g $m_{AR} = m_{AC}$

$\frac{q-1}{-2+1} = \frac{1-0}{-2-5}$

$\frac{q-1}{-1} = \frac{1}{-7}$

$\therefore q = \frac{8}{7}$

2 a Substitute $x = 1$ and $y = -3$ in LHS. If LHS=0, then the point $N(1; -3)$ lies on the circle.

$LHS = x^2 + 4x + y^2 + 2y - 8$

$= (1)^2 + 4(1) + (-3)^2 + 2(-3) - 8 = 0$

$\therefore N$ lies on the circle

b First determine the centre of the circle:

$x^2 + 4x + 4 + y^2 + 2y + 1 = 8 + 4 + 1$

$(x + 2)^2 + (y + 1)^2 = 13$

Centre of circle is $M(-2; -1)$

$m_{MN} = \frac{-1+3}{-2-1} = -\frac{2}{3}$

MN⊥PN (radius⊥tangent)

$\therefore m_{PN} = \frac{3}{2}$

Substitute $N(1; -3)$: $y = \frac{3}{2}x + c$

$-3 = \frac{3}{2}(1) + c \qquad \therefore c = -\frac{9}{2}$

$y = \frac{3}{2}x - \frac{9}{2}$

c $\theta = tan^{-1}\left(\frac{3}{2}\right) = 56,3°$

d x −intercept where $y = 0$:

$0 = \frac{3}{2}x - \frac{9}{2} \qquad \therefore x = 3$

e y −intercepts are where $x = 0$:

$(0)^2 + 4(0) + y^2 + 2y - 8 = 0$

$\therefore y^2 + 2y - 8 = 0$

$(y + 4)(y - 2) = 0$

The points are $(0; -4)$ and $(0; 2)$.

3 a $m_{RO} = \frac{-12}{-6} = 2$

b PS⊥RN (RN is altitude of Δ)

$m_{PS} \times m_{RN} = -1$

$\therefore m_{PS} = -\frac{1}{2}$

c $P(0;6)$ (y −intercept of PR)

$\therefore y = -\frac{1}{2}x + 6$

d $tan^{-1}\left(\frac{1}{2}\right) = 26{,}57°$

Inclination of PS $= 180° - 26{,}57° = 153{,}43°$

e Substitute $N(2n; 3\frac{3}{5} + n)$ into equation of PS

$3\frac{3}{5} + n = -\frac{1}{2}(2n) + 6$

$3\frac{3}{5} + n = -n + 6$

$2n = 2\frac{2}{5} = \frac{12}{5}$

$n = \frac{6}{5}$

f Find equation of SM. SM is the median, so M is the midpoint of PR.

$M\left(\frac{-6+0}{2}; \frac{-12+6}{2}\right) = (-3; -3)$

$m_{MS} = 1$ so equation of SM: $y = x$

Solve equations of SM and PS simultaneously to calculate coordinates of S

$x = -\frac{1}{2}x + 6$ $\therefore x = 4; y = 4$

$S(4;4)$

4 a $x^2 + 4x + y^2 - 2y = 4$

$x^2 + 4x + 4 + y^2 - 2y + 1 = 4 + 4 + 1$

$(x+2)^2 + (y-1)^2 = 9$

Centre $M(-2;1)$ radius$= 3$

b Substitute $N(p;1)$ into equation of circle.

$p^2 + 1^2 + 4(p) - 2(1) - 4 = 0$

$p^2 + 4p - 5 = 0$

$(p+5)(p-1) = 0$

$\therefore p = 1$ as $p > 0$

c Radius through N is horizontal.

Therefore the tangent will be vertical.

Equation of tangent: $x = 1$

5 a $m_{AD} = \frac{3-0}{-3-0} = -1$

AD goes through origin: So, equation is $y = -x$

b $BD^2 = DC^2$

$(x-2)^2 + (y-3)^2 = (x-6)^2 + (y+1)^2$

Substitute $y = -x$

$(x-2)^2 + (-x-3)^2 = (x-6)^2 + (-x+1)^2$

$x^2 - 4x + 4 + x^2 + 6x + 9 = x^2 - 12x + 36 + x^2 - 2x + 1$

$16x = 24$

$x = \frac{3}{2} \qquad \therefore y = -\frac{3}{2}$

c $m_{BD} = \frac{3-(-\frac{3}{2})}{2-\frac{3}{2}} = 9$

Substitute $B(2;3)$ into $y = 9x + c$

$3 = 9(2) + c \qquad \therefore c = -15$

$y = 9x - 15$

d Inclination of BD= $tan^{-1}(9) = 83{,}7°$

$m_{BC} = \frac{3-(-1)}{2-6} = -1$

Inclination of BC= $135°$

$\therefore \theta = 135° - 83{,}7° = 51{,}3°$

e $BD = \sqrt{\left(2-\frac{3}{2}\right)^2 + \left(3+\frac{3}{2}\right)^2} = \frac{\sqrt{82}}{2}$

$BC = \sqrt{(3+1)^2 + (2-6)^2} = 4\sqrt{2}$

$Area\ of\ \Delta BDC = \frac{1}{2} BD \times BC \times sin\theta$

$= \frac{1}{2} \times \frac{\sqrt{82}}{2} \times 4\sqrt{2} \times sin51{,}3°$

$= 10\ sq\ units$

6 a First determine equation of AC

$m_{AC} = \frac{3-(-3)}{2-5} = -2$

Substitute $(2;3)$: $3 = -2(2) + c \qquad \therefore y = -2x + 7$

$x - intercept(y = 0): x = \frac{7}{2} \qquad D\left(\frac{7}{2};0\right)$

b $BC^2 = AC^2$

$(p-5)^2 + (0+3)^2 = (5-2)^2 + (-3-3)^2$

$p^2 - 10p + 25 = 9 + 36$

$p^2 - 10p - 20 = 0$

$p = \frac{10 \pm \sqrt{180}}{2} = 5 \pm 3\sqrt{5}$

$p = 5 - 3\sqrt{5}$

c $m_{AC} = -2$

$Inclination\ of\ AC = 180° - tan^{-1}(2) = 116{,}6°$

d $B(-1;0)$

$m_{AB} = \frac{3-0}{2+1} = 1$

Inclination of $AB = 45°$

$\hat{A} = inclination\ of\ AC - inclination\ of\ AB$

$= 116{,}6° - 45°$

$= 71{,}6°$

7 The line will be a tangent if it intersects the circle in only one point.

Substitute $y = x + 1$ into equation of circle and solve for x.

There should be only one solution.

$x^2 + (x+1)^2 + 6(x+1) - 7 = 0$

$x^2 + x^2 + 2x + 1 + 6x + 6 - 7 = 0$

$2x^2 + 8x = 0$

$x = 0$ or $x = -4$

The line is NOT a tangent.

8 a $y = 2$ at C. Substitute into $3x + 4y + 7 = 0$

$3x + 4(2) + 7 = 0$

$3x = -15$

$x = -15$

$\therefore C(-5;2)$ and the radius is 5.

$(x+5)^2 + (y-2)^2 = 25$

b length of $DE = 10$

c $m_{PE} = \frac{2+1}{0+1} = 3$

$m_{perp\ bisector} = -\frac{1}{3}$

Midpoint of PE= $\left(\frac{0-1}{2};\frac{2-1}{2}\right) = \left(-\frac{1}{2};\frac{1}{2}\right)$

Substitute midpoint into $y = -\frac{1}{3}x + c$

$\frac{1}{2} = -\frac{1}{3}\left(-\frac{1}{2}\right) + c$

$c = \frac{1}{3}$

$y = -\frac{1}{3}x + \frac{1}{3}$

d $3x + 4\left(-\frac{1}{3}x + \frac{1}{3}\right) + 7 = 0$

$3x - \frac{4}{3}x + \frac{4}{3} + 7 = 0$

$\frac{5}{3}x = -\frac{25}{3}$

$x = -5$

$y = -\frac{1}{3}(-5) + \frac{1}{3} = 2$ The lines intersect at $(-5;2)$

9 a Let the coordinates of S be $(x; 0)$

ST$\perp$SR

$m_{ST} \times m_{SR} = \frac{4}{-x} \times \frac{4}{x-4} = -1$

$x(x-4) = 16$

$x^2 - 4x - 16 = 0$

$x = \frac{4 \pm \sqrt{(-4)^2 - 4(-16)}}{2} = 2 \pm 2\sqrt{5}$

But S is on positive x −axis, so $S\left(2 + 2\sqrt{5}; 0\right)$

b $m_{ST} = \frac{4-0}{0-\left(2+2\sqrt{5}\right)} = -0{,}62$

c Inclination of TS$= 180° - tan^{-1}(0{,}62) = 148{,}20°$

$m_{TR} = \frac{4+4}{0-4} = -2$

Inclination of TR$= 180° - tan^{-1}(2) = 116{,}57°$

$R\hat{T}S = 148{,}20° - 116{,}57° = 31{,}63°$

10 a $x^{\,2} + y^2 - 4x + 6y + 3 = 0$

$x^{\,2} - 4x + y^2 + 6y = -3$

$x^{\,2} - 4x + 4 + y^2 + 6y + 9 = -3 + 4 + 9$

$(x-2)^2 + (y+3)^2 = 10$

Centre is $(2; -3)$

$m_{radius} = \frac{-2+3}{5-2} = \frac{1}{3}$

$m_{tangent} = -3$

Substitute $(5; -2)$ into $y = -3x + c$

$-2 = -3(5) + c$

$c = 13$

$\therefore y = -3x + 13$

b $\sqrt{(x-2)^2 + (y+3)^2} = \sqrt{20}$

$(x-2)^2 + (y+3)^2 = 20$

Substitute $y = -3x + 13$ into equation above:

$(x-2)^2 + (-3x + 13 + 3)^2 = 20$

$(x-2)^2 + (-3x + 16)^2 = 20$

$x^2 - 4x + 4 + 9x^2 - 96x + 256 = 20$

$10x^2 - 100x + 240 = 0$

$x^2 - 10x + 24 = 0$

$(x-6)(x-4) = 0$

$x = 6$ or $x = 4$

$y = -3(6) + 13 = -5$ or $y = -3(4) + 13 = 1$

$T(6; -5)$ or $T(4; 1)$

Chapter 10: Euclidian geometry

1 a $\hat{B}_1 = \hat{M}_1$ tan chord
$\hat{B}_1 = \hat{C}$
$\therefore \hat{M}_1 = \hat{C}$
$\therefore MN||CA$ corr ∠s
b $\hat{K}_1 = \hat{M}_2$ alt ∠s
$\hat{K}_1 = \hat{N}_2$ tan chord
$\therefore \Delta KMN$ is isosceles
c $\hat{K}_4 = \hat{N}_2$ alt ∠s
$\hat{N}_2 = \hat{B}_3$ ∠s in same segment
$\hat{A}_3 = \hat{B}_3$ ∠s in same segment
$\therefore \hat{K}_4 = \hat{A}_3$
$\therefore NK||AP$ alt ∠s=
$\therefore \frac{BN}{NA} = \frac{BK}{KP}$ line || to one side of Δ
But $\frac{BN}{NA} = \frac{BM}{MC}$ line || to one side of Δ
$\therefore \frac{BK}{KP} = \frac{BM}{MC}$
d $\hat{A}_3 = \hat{B}_3$ ∠s in same segment
$\hat{B}_3 = \hat{B}_2$ equal chords subt equal ∠s
$\therefore \hat{A}_3 = \hat{B}_2$
$\therefore DA$ is a tangent to the circle through A, B and K

2 a $\hat{C}_3 = C\hat{P}R$ ∠s opp equal sides
$\hat{C}_3 + \hat{C}_2 = \hat{A}_1 + \hat{B}$ ext ∠ of Δ
$\hat{C}_2 = \hat{B}$ tan chord
$\therefore \hat{C}_3 = \hat{A}_1$
$\therefore \hat{A}_1 = C\hat{P}R$ both $= \hat{C}_3$
ACPR is a cyclic quadrilateral (ext ∠ of quad)
b In ΔCBA and ΔRPA:
$\hat{P}_2 = \hat{C}_2$ ∠s in same segment
$= \hat{B}$ proven in 2 a
$\therefore \hat{B} = \hat{P}_2$
$\hat{C}_1 = A\hat{R}P$ ext ∠ of cyclic quad
$\hat{A}_1 = \hat{A}_3$ 3rd ∠ of Δ
$\therefore \Delta CBA|||\Delta RPA$ ∠∠∠

c $\frac{RP}{CB} = \frac{RA}{CA}$ from 2 b

$RP = \frac{CB.RA}{CA}$ but $RP = RC$

$\therefore RC = \frac{CB.RA}{CA}$

d In ΔRAC and ΔRCB:

$\hat{C}_2 = \hat{B}$ tan chord

$\hat{R}_1$ is common

$R\hat{C}B = R\hat{A}C$ 3rd angle

$\therefore \Delta RAC|||\Delta RCB$ $\angle\angle\angle$

$\frac{AC}{CB} = \frac{RC}{RB}$ Δs |||

$RB.AC = RC.CB$

e $\frac{CB}{RP} = \frac{CA}{RA}$ from 2 b

$\frac{CB}{RC} = \frac{CA}{RA}$ RC=RP

$AC = \frac{CB.RA}{RC}$ ...(i)

From 2 d $AC = \frac{RC.CB}{RB}$

$\therefore \frac{CB.RA}{RC} = \frac{RC.CB}{RB}$

$\therefore RC^2 = RA.RB$

3 a $\hat{B}_2 = \hat{A}_3 = x$ $\angle$s opp equal sides

$\widehat{M}_1 = 180° - 2x$ sum $\angle$s of Δ

$\therefore \widehat{D} = 2x$

b i $\hat{C} = \frac{\widehat{M}_1}{2}$ $\angle$ at centre =2x$\angle$circ

$= 90° - x$

$C\hat{B}D = 180° - (90° - x + 2x)$ sum $\angle$s of Δ

$= 90° - x$

$\widehat{N}_1 = \hat{C} = 90° - x$ ext $\angle$ of cyclic quad

$\therefore C\hat{B}D = \widehat{N}_1$

$\therefore CB||AN$ corr $\angle$s

b ii $C\hat{B}A = \widehat{D} = 2x$ tan chord

$C\hat{B}A = \hat{A}_2$ alt $\angle$s

$\hat{A}_2 = \widehat{D}$

$\therefore$AB is a tangent ($\angle$betw line&chord= $\angle$sub chord)

4 a $\hat{B}_3 = \hat{E}_1 = x$ ∠s in same segment
$\hat{B}_3 = \hat{D}_2 = x$ ∠s opp = sides
$B\hat{O}D = 180° - 2x$ sum ∠s of Δ
$\hat{A} = 90° - x$ ∠ at centre =2x∠circ
b i $\hat{C}_1 = 90° - x$ ext ∠ of cyclic quad
$\hat{F}_2 = 180° - (x + 90° - x)$ sum ∠s of Δ
$= 90°$
In ΔBEF and ΔCEF:
$\hat{F}_1 = \hat{F}_2 = 90°$ adj ∠s str line
BF = FC
FE is common
$\Delta BEF \equiv \Delta CEF$ s∠s
BE = EC (≡)
ii $\hat{B}_1 = 90° - x$ sum ∠s of Δ
$\therefore \hat{B}_1 = \hat{A}$
∴ BE is not a tangent $\left(\hat{B}_1 + \hat{B}_2 \neq \hat{A}\right)$

5 a P is midpoint of AC medians concur
AB||PM midpt theorem
In ΔBNC:
$\frac{ND}{NC} = \frac{BM}{BC} = \frac{AP}{AC}$ line || 1 side of Δ
$= \frac{BM}{2BM} = \frac{1}{2}$
b In ΔAMP:
$\frac{AO}{OM} = \frac{2OM}{OM}$
$\frac{RP}{PC} = \frac{RP}{AP}$ BP is a median
$= \frac{OM}{AM}$ line || 1 side of Δ
$= \frac{OM}{3OM}$
$= \frac{1}{3}$

6 a $\hat{C}_2 = 90°$ ∠ in semi ⊙
$\hat{M}_2 = 90°$ AM⊥NM
$\therefore NQ||CD$ corr ∠s=
b $\hat{C}_1 = \hat{N}$ || lines, corr ∠s
$\hat{A}_2 = \hat{C}_1$ tan chord
$= \hat{N}$
∴ ANCQ is a cyclic quad ∠s subt by same line segm

c i In ΔPCD and ΔPAC:

$\hat{C}_1 = \hat{A}_2$ tan chord

$\hat{P}$ is common

$\hat{D}_1 = A\hat{C}P$ 3rd $\angle$

$\therefore \Delta PCD \,|||\, \Delta PAC \;\; \angle\angle\angle$

ii $PC^2 = AP.DP$

d In ΔNBC and ΔBCD:

$\hat{N} = \hat{A}_2$ $\angle$s in same segm

$= \hat{B}_2$ $\angle$s in same segm

$\hat{C}_4 = \hat{A}_1$ tan chord

$= \hat{D}_2$ $\angle$s in same segm

$\hat{B}_1 = B\hat{C}D$ 3rd $\angle$

$\therefore \Delta NBC \equiv \Delta BCD \;\; \angle\angle\angle$

$\therefore \frac{BC}{NB} = \frac{CD}{NB}$

$BC^2 = CD.NB$

e $1 - \frac{BM^2}{BC^2} = \frac{BC^2 - BM^2}{BC^2}$

$= \frac{MC^2}{BC^2}$ Pyth.

$= \frac{PC^2}{BC^2}$

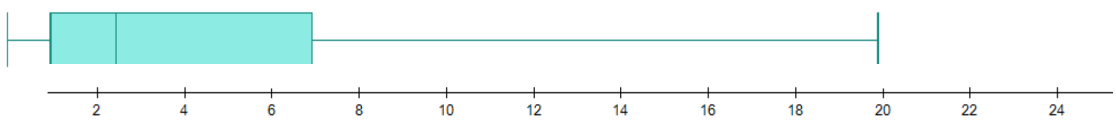
$= \frac{AP.DP}{CD.NB}$

Chapter 11: Statistics: regression and correlation

1 a

	Lower Q	Median	Upper Q
Matches played	3	5	6
Wins	1	7	3
Goals scored against	3	4,5	9

b

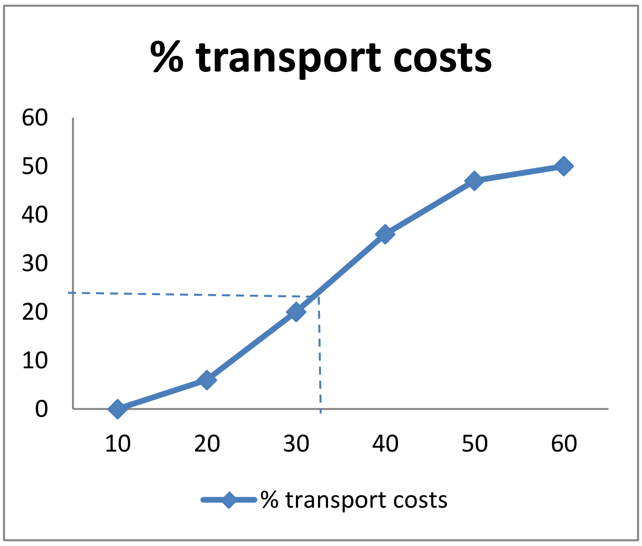


Positively skewed (skewed to the right)

c $\frac{65}{14} = 4{,}64$

d Standard deviation $= 1{,}72$

2 a



b Median $= \pm 32\%$

c

Class midpoint	Frequency	FreqxMidpoint
15	6	90
25	14	350
35	16	560
45	11	495
55	3	165
TOTAL	50	1660

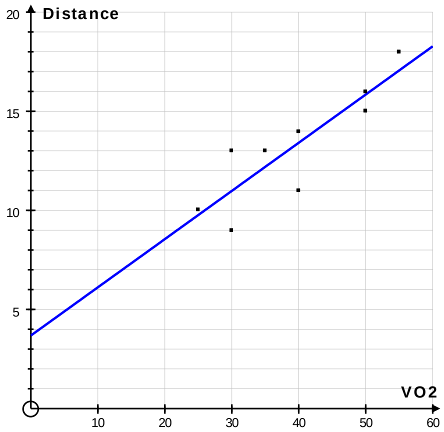
Estimated mean $\bar{x} = \frac{1660}{50} = 33{,}2\%$

d

Class midpt x_i	Freq f	$\bar{x} - x_i$	$(\bar{x} - x_i)^2$	$f(\bar{x} - x_i)^2$
15	6	-18,2	331,24	1987,44
25	14	-8,2	67,24	941,36
35	16	1,8	3,24	51,84
45	11	11,8	139,24	1531,64
55	3	21,8	475,24	1425,72
TOTAL	50			5938

Standard deviation$= \sqrt{\frac{5938}{50}} = 10{,}90$

3 a & c



b $y = 0{,}2432x + 3{,}6834$

d Substitute $y = 19$ then $x = 62{,}98$ (VO^2)

e $r = 0{,}8985\ldots$

Strong positive correlation

4

$Number$	$(Number - mean)^2$
4	36
8	4
10	0
x	$(x - 10)^2$
y	$(y - 10)^2$

Mean= 10

$\therefore \frac{4+8+10+x+y}{5} = 10$

Which simplifies to: $x + y = 28 \ldots.(1)$

Standard deviation = 4

$\therefore \sqrt{\frac{36+4+0+(x-10)^2+(y-10)^2}{5}} = 4$

Which simplifies to: $(x - 10)^2 + (y - 10)^2 = 40 \ldots (2)$

Substitute $y = 28 - x$ from (1) into (2):

$x^2 - 28x + 192 = 0$

$(x - 12)(x - 16) = 0$

$x = 12$ or $x = 16$

$y = 16$ or $y = 12$

5 The standard deviation will remain 7,5.
If all the numbers are 2 bigger, then the mean will also be 2 bigger.
The difference between each number and the mean will therefore remain the same leaving the standard deviation unchanged.

Chapter 12: Probability

1 7 spaces that have to be filled using 7 digits without repetition(as 0,7 and 4 may not be used again)
$\therefore 7! = 5040$

2 7 spaces have to be filled – 10 digits are available for each space
$\therefore 10^7 = 10\,000\,000$

3 P(Queen of diamonds)= $\frac{1}{52}$

4 a $11!$
b $\frac{11!}{2!2!2!2!2!} = 1\,247\,400$ (5 letters repeat)

5 a Regard the 4 English books as a unit. The number of arrangements for the English books is 4!=24
Total number of arrangements= $4! \times 6! = 17\,280$
b $4! \times 3! \times 2! \times 3! = 1728$
c $9! = 362\,880$

6 $12 \times 11 = 132$

7 First calculate the total number of words: $\frac{11!}{2!2!2!} = 4\,989\,600$
Now calculate how many of these WILL start and end on the same letter.
It can start and end with M, A or T
$\therefore \frac{9!}{2!2!} = 90720$
P(not start and end on same letter)= $1 - \frac{90\,720}{4\,989\,600} = \frac{54}{55}$

8 a $\frac{10! \times 2}{2!2!2!} = 907\,200$ b $\frac{10!}{2!2!2!} = 453\,600$

Exemplar Paper 1 (3 hours; 150 marks)

1 a Solve for x:

i $x + 2 = \frac{2}{x+1}$ (4)

ii $x - \sqrt{x} = 6$ (4)

iii $\frac{(x^2+4)(2-x)}{x+2} \geq 0$ (6)

iv $5^{x-2} + 5^{x+1} = 126$ (5)

b Consider the equation: $f(x) = 2x^3 + px^2 + bx - 9p$

If $(2x + p)$ is a factor of $f(x)$ and $p \neq 0$, determine the value(s) of b. (5)

c 2 is a root of $2x^2 - 3x - p = 0$. Determine the value of p and hence the other root. (4)

[28]

2 a The sum of the first 20 terms of an arithmetic progression is 410, while the sum of the next 30 terms is 2865. Determine the first three terms of the progression. (7)

b $3; x; 15; y; 35$ is a quadratic sequence.

i Determine the values if x and y. (4)

ii Determine formula for T_n. (4)

c Find n such that $\sum_{k=7}^{n}(2k - 3)$ is equal to the sum of the first 6 terms of the sequence $-24; 48;\ -96; \ldots$ (7)

d For which value(s) of x will the following series be convergent?

$(x + 2) + (x + 2)^2 + (x + 2)^3 + \cdots$ (2)

[24]

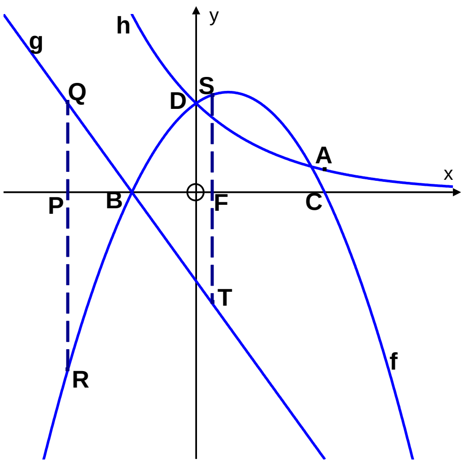
3 a Melissa decides to save R1 200 per month for a certain period. The bank offers her an interest rate of 12% p.a. compounded monthly for this period. Determine how long Melissa has to make this monthly payment if she wants to have a lump sum of R200 000. (5)

b Richard plans to buy a house on a 20 year mortgage and can only afford to pay R5 000 per month. If the interest rate is currently 12% per annum compounded monthly, determine the size of the mortgage he can take, if he starts paying one month after the mortgage was approved. (3)

c An amount of R300 000 is to be used to provide quarterly withdrawals for the next 10 years. The withdrawal amount will remain fixed and the first withdrawal will be in 3 months' time. An interest rate of 15% p.a. compounded quarterly applies. Determine the value of each quarterly withdrawal. (4)

[12]

4 In the diagram f is the graph of $y = -\frac{1}{2}x^2 + \frac{1}{2}x + k$ cuts the x –axis at B and C and the y –axis at D. g is the graph of $y = ax - \frac{3}{2}$ and cuts the x –axis at B. h is the graph of $y = m^x$ and cuts the y –axis at D. QR and ST are parallel to the y –axis. $A\left(x; \frac{1}{4}\right)$ is a point on h and vertically above C.



a Determine the values of k and m. (6)

b Determine the value of a. (2)

c Calculate the length of QR if OP = 2 units. (4)

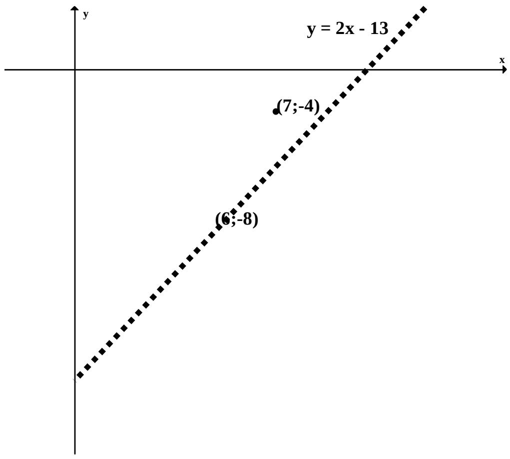
d Determine the length op OF, if ST = 4 units. (4)

e Determine the equation of h^{-1}. (2)

f Write down the domain of h^{-1}. (2)

[20]

5 The functions $f(x) = \frac{a}{x+b} + c$ and $g(x) = 2x - 13$ intersect each other. The asymptotes of $f(x)$ intersect in $(6; -8)$. $f(x)$ goes through $(7; -4)$.



a Determine the values of a, b and c. (4)

b Determine the co-ordinates of the intersects of f and g. (5)

c For which values of x would $g(x) \geq f(x)$? (3)

d Determine the equation of the dotted line which is the axis of symmetry of the hyperbola. (3)

[15]

6 Determine:

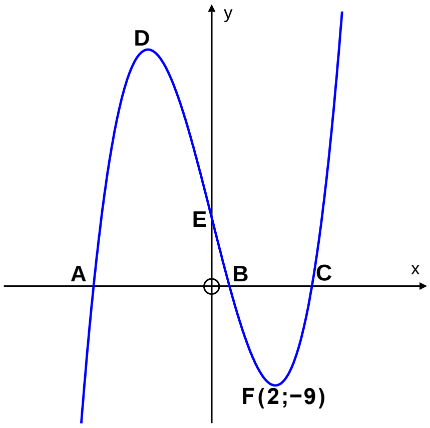
a $\lim_{x \to 1} \frac{x^2-1}{1-x}$ (3)

b $f'(x)$ from first principles if $f(x) = -2x^2$. (4)

c $g'(t)$ if $g(t) = 2\sqrt{t} + \frac{1}{2t^2}; t \neq 0$ (4)

[11]

7 The figure shows the graph of $f(x) = 2x^3 + ax^2 + bx + 3$. The curve has a local minimum turning point F at $(2; -9)$.



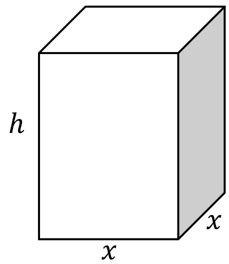
a Show that $a = -5$ and $b = -4$. (6)

b If it is given that $A(-1; 0)$, calculate the coordinates of B and C. (5)

c Determine the equation of the tangent to the graph at $x = 3$. (4)

[15]

8 A container firm is designing an open-top rectangular box that will hold $108\ cm^3$. The box has a square base with sides x and height h.



a Show that the total outside surface area of the box will be $S = x^2 + \frac{432}{x}$. (4)

b For which value of x and h will the outer surface area be a minimum. (5)

[9]

9 a A six-member working group is to be selected from five teachers and nine students. If the working group is randomly selected, what is the probability that it will include at least two teachers? (4)

b $P(A \text{ or } B) = 0{,}6$ and $P(A) = 0{,}2$

i Find $P(B)$ given that events A and B are mutually exclusive. (2)

ii Find $P(B)$ given that events A and B are independent. (4)

c i In how many ways can the letters of the word PROBABILITY be arranged to form different "words" – the word "probability" itself is included? (3)

ii In how many ways can the letters of the word PROBABILITY be arranged to form different "words" if the R and O have to be kept together? (3)

[16]

MEMORANDUM: Exemplar Paper 1

1 a i $x + 2 = \frac{2}{x+1}$

$(x+2)(x+1) = 2$

$x^2 + 3x = 0$

$x(x+3) = 0$

$\therefore x = 0 \text{ or } x = -3$

ii $x - \sqrt{x} = 6$

Let $= \sqrt{x}$, then $k^2 = x$

$k^2 - k - 6 = 0$

$(k-3)(k+2) = 0$

$k = \sqrt{x} = 3$ or $k = \sqrt{x} = -2$

$x = 9$ Not valid

iii $\frac{(x^2+4)(2-x)}{x+2} \geq 0$

$(x^2 + 4) > 0$ for all values of $x \in R$

$\frac{(2-x)}{(x+2)} \geq 0$

$\frac{(x-2)}{(x+2)} \leq 0$

$\therefore -2 < x \leq 2$

iv $5^{x-2} + 5^{x+1} = 126$

$5^x.5^{-2} + 5^x.5^1 = 126$

$5^x\left(\frac{1}{25} + 5\right) = 126$

$5^x\left(\frac{126}{25}\right) = 126$

$5^x = 5^2$

$x = 2$

b If $(2x + p)$ is a factor, then $f\left(-\frac{p}{2}\right) = 2\left(\frac{-p}{2}\right)^3 + p\left(-\frac{p}{2}\right)^2 + b\left(-\frac{p}{2}\right) - 9p = 0$

$-\frac{p^3}{4} + \frac{p^3}{4} - \frac{bp}{2} - 9p = 0$

$\times 2)\ \ bp = -18p$

$\div p)\ \ b = -18$

c Substitute $x = 2$: $2(2)^2 - 3(2) - p = 0$

$\therefore p = 2$

$2x^2 - 3x - 2 = 0$

$(2x + 1)(x - 2) = 0$

$\therefore x = -\frac{1}{2}$ is the other root

2 a $S_{20} = 410$

$S_{50} = S_{20} + sum\ of\ next\ 30\ terms = 410 + 2865 = 3275$

$410 = \frac{20}{2}[2a + 19d]$

$41 = 2a + 19d \ldots .(1)$

$3275 = \frac{50}{2}[2a + 49d]$

$131 = 2a + 49d \ldots .(2)$

(2)-(1): $30d = 90$

$\therefore d = 3$ en $a = -8$

b i $x = 8\,; y = 24$

ii T: $3\,;\ 8\,;\ 15\,;\ 24\,;\ 35$

f: 5 ; 7 ; 9 ; 11

s: 2 ; 2 ; 2

$a = 2 \div 2 = 1$ $b = 5 - 3(1) = 2$ $c = 3 - 1 - 2 = 0$

$T_n = n^2 + 2n$

c $-24; 48; -96; \ldots$ is a geometric series with $a = -24$ and $r = -2$

$S_6 = \frac{-24((-2)^6-1)}{-2-1} = 504$

$\sum_{k=7}^{n}(2k-3) = 11 + 13 + 15 + \cdots + (2n-3)$

$504 = \frac{n}{2}[2(11) + (n-1)(2)]$

$n^2 + 10n - 504 = 0$

$(n+28)(n-18) = 0$

$\therefore n = 18$

d $r = x + 2$

For convergent series $-1 < r < 1$

$-1 < x + 2 < 1$

$-3 < x < -1$

3 a $200\,000 = \frac{1200[\left(1+\frac{0,12}{12}\right)^n - 1]}{\frac{0,12}{12}}$

$(1,01)^n = \frac{5}{3}$

$n = \frac{log\frac{5}{3}}{log1,01} = 51,33755$

Melissa must make at least 52 payments

b $P = \frac{5000\left[1-\left(1+\frac{0,12}{12}\right)^{-240}\right]}{\frac{0,12}{12}} = R454\,097,08$

c $300000 = \frac{x\left[1-\left(1+\frac{0,15}{4}\right)^{-40}\right]}{\frac{0,15}{4}}$

$x = \frac{300000\times\frac{0,15}{4}}{\left[1-\left(1+\frac{0,15}{4}\right)^{-40}\right]} = R14957,84$

4 a D is the y –intercept of f and h .

Substitute $x = 0$ into $y = m^x$

$\therefore y = 1$

$\therefore k = 1$

To find m we need the coordinates of A.

First find the roots of f os we can get the x-value of A.

$-\frac{1}{2}x^2 + \frac{1}{2}x + 1 = 0$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2$ at C and A

Sub $A\left(2;\frac{1}{4}\right)$ into $y = m^x$

$\frac{1}{4} = m^2$

$\therefore m = \frac{1}{2}$

b Sub $B(-1;0)$: $0 = a(-1) - \frac{3}{2}$ $\therefore a = -\frac{3}{2}$

c $x = -2$ at Q and R

$QR = y_Q - y_R$

$= -\frac{3}{2}(-2) - \frac{3}{2} - [-\frac{1}{2}(-2)^2 + \frac{1}{2}(-2) + 1]$

$= \frac{7}{2}$

d $-\frac{1}{2}x^2 + \frac{1}{2}x + 1 + \frac{3}{2}x + \frac{3}{2} = 4$

$-\frac{1}{2}x^2 + 2x - \frac{3}{2} = 0$

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$\therefore OF = 1$

e $h^{-1} = \log_{\frac{1}{2}} x$

f $x > 0; x \in R$

5 a Asymptotes go through $(6;-8)$

$\therefore b = -6$ and $c = -8$

Substitute $(7;-4)$ into $f(x) = \frac{a}{x-6} - 8$

$-4 = \frac{a}{7-6} - 8$

$\therefore a = 4$

b $\frac{4}{x-6} - 8 = 2x - 13$

$\frac{4}{x-6} = 2x - 5$

$4 = (2x-5)(x-6)$

$2x^2 - 17x + 26 = 0$

$(2x-13)(x-2) = 0$

$x = \frac{13}{2}$ or $x = 2$

$y = 0$ or $y = -9$

Intersects are $\left(\frac{13}{2};0\right)$ and $(2;-9)$

c $x \in [2;6)$ or $x \in [\frac{13}{2};\infty)$

d Substitute $(6;-8)$ into $y = x + c$

$-8 = 6 + c$ $\therefore c = -14$ $y = x - 14$

6 a $\lim_{x\to 1}\frac{x^2-1}{1-x} = \lim_{x\to 1}\frac{(x-1)(x+1)}{-(x-1)}$

$= \lim_{x\to 1} -(x+1) = -2$

b $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0}\frac{-2(x+h)^2-(-2x^2)}{h}$

$= \lim_{h\to 0}\frac{-2xh-2h^2}{h}$

$= \lim_{h\to 0} - 2x - h$

$= -2x$

c $g(t) = 2\sqrt{t} + \frac{1}{2t^2} = 2t^{\frac{1}{2}} + \frac{1}{2}t^{-2}$

$g'(t) = 2 \times \frac{1}{2}t^{-\frac{1}{2}} + \frac{1}{2} \times -2t^{-3}$

$= \frac{1}{\sqrt{t}} - \frac{1}{t^3}$

7 a $f(2) = -9$ and $f'(2) = 0$

$f'(x) = 6x^2 + 2ax + b$

$0 = 6(2)^2 + 2a(2) + b$

$4a + b = -24 \ldots.(1)$

$-9 = 2(2)^3 + a(2)^2 + b(2) + 3$

$2a + b = -14$...(2)

(1)-(2): $2a = -10$

$\therefore a = -5$

$b = -2a - 14$

$= -2(-5) - 14 = -4$

b From (a) it follows that $f(x) = 2x^3 - 5x^2 - 4x + 3$

If $x = -1$ is a root, then $(x+1)$ is a factor of f

$f(x) = 2x^3 - 5x^2 - 4x + 3$

$= (x+1)(2x^2 - 7x + 3$

$= (x+1)(2x-1)(x-3)$

$x = -1$, or $x = \frac{1}{2}$ or $x = 3$

$B\left(\frac{1}{2}; 0\right)$ and $C(3; 0)$

c $f'(x) = 6x^2 - 10x - 4$

$f'(3) = 6(3)^2 - 10(3) - 4 = 20$

Sub $(3; 0)$ into $y = 20x + c$

Eq of tangent: $y = 20x - 60$

8 a Volume $= 108$

$x^2h = 108$

$\therefore h = \frac{108}{x^2}$

$S = x^2 + 4xh$

$= x^2 + 4x\left(\frac{108}{x^2}\right)$

$= x^2 + \frac{432}{x}$

b S will be a minimum where $S'(x) = 0$

$S = x^2 + 432x^{-1}$

$S'(x) = 2x - \frac{432}{x^2} = 0$

$x^3 = 216$

$x = 6\ m$ and $h = \frac{108}{(6)^2} = 3\ m$

9 a Total number of different six-member groups$= \frac{14!}{7!} = 17\ 297\ 280$

Number of groups with no teacher$= \frac{9!}{3!} = 60\ 480$

Number of groups with one teacher only$= 5 \times 9 \times 8 \times 7 \times 6 \times 5 = 75\ 600$

Total number of groups with less than two teachers$= 136\ 080$

Total number of groups with two or more teachers

$= 17\ 297\ 280 - 136\ 080 = 17\ 161\ 200$

P(two or more teachers)$= \frac{17\ 161\ 200}{17297280} = 0{,}99$

b i $P(AorB) = P(A) + P(B)$ for mutually exclusive

$0{,}6 = 0{,}2 + P(B)$

$P(B) = 0{,}4$

ii $P(A\ and\ B) = P(A) \times P(B)$ for independentevents

$P(A\ or\ B) = P(A) + P(B) - P(A\ and\ B)$

$P(A\ or\ B) = P(A) + P(B) - P(A) \times P(B)$

$0{,}6 = 0{,}2 + P(B) - 0{,}2P(B)$

$0{,}4 = 0{,}8P(B)$

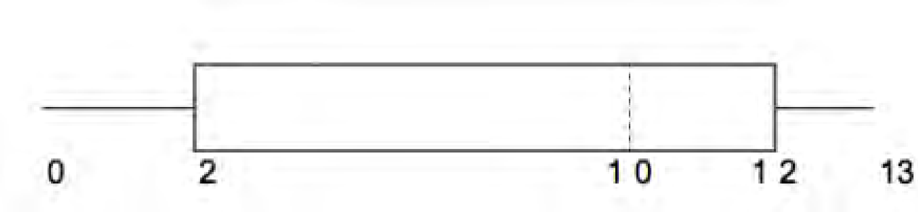
$P(B) = 0{,}5$

c i $\frac{11!}{2!2!} = 9\ 979\ 200$

ii $\frac{10!}{2!2!} = 907\ 200$

Exemplar Paper 2 (3 hours; 150 marks)

1 Given the following box-and-whisker plot:



a Which quarter has the smallest spread of data?
What is the spread? (2)

b Determine the inter quartile range. (2)

c Are there more data in the interval 5-10 or in the interval 10-13?
How do you know this? (2)

d Which interval has the fewest data in it? Is it 0-2, 2-4, 10-12 or 12-13? How do you know it? (2)

[8]

2 A factory produces and stockpiles metal sheets to be shipped to a motor vehicle manufacturing plant. The factory only ships when there is a minimum of 3254 sheets in stock at the beginning of that day. The table shows the day and the number of sheets in stock at the beginning of that day.

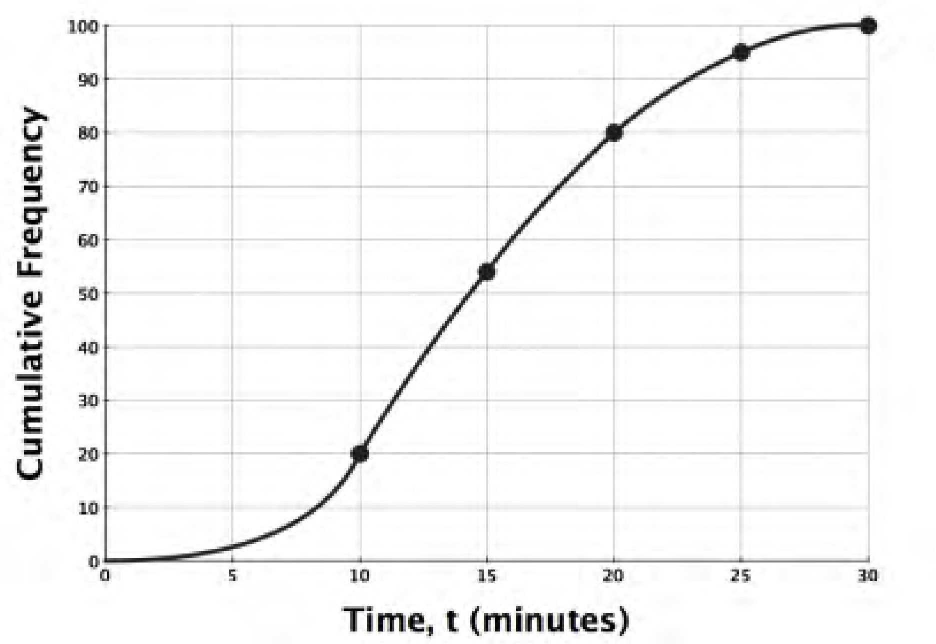
Day	1	2	3	4	5	6
Sheets	854	985	1054	1195	1204	1384

a Determine the equation of the least squares regression line for this set of data rounding coefficients to three decimal places. (3)

b Use this equation to determine the day the sheets will be shipped. (3)

[6]

3 The ogive below represents the results of a survey amongst first year students on the average time per day they spend exercising. Answer the questions that follow.



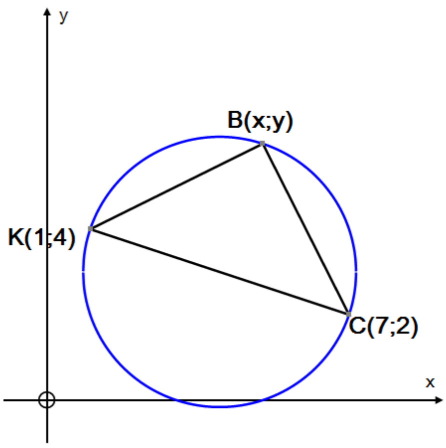
a How many students participated in the survey? (1)

b Approximately how many students spend more between 10 and 20 minutes per day exercising? (1)

c Use the ogive to determine the median time spent on daily exercise. (2)

[4]

4 In the diagram, KC is a diameter of the circle and $K(1;4)$; $C(7;2)$ and $B(x;y)$ are points on the circle.



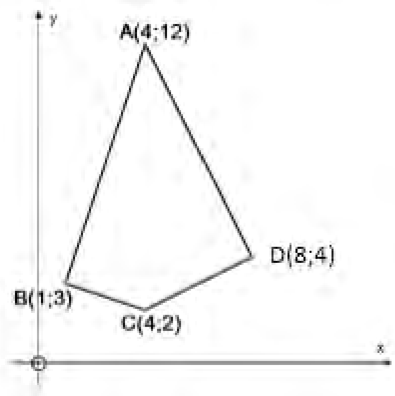
Determine:

a the equation of the circle (5)

b point B if the gradient of KB$= \frac{1}{2}$ (9)

[14]

5 In the diagram, ABCD is a quadrilateral with $A(4;12)$, $B(1;3)$, $C(4;2)$ and D(8;4).



a Determine the gradients of BC and CD. (4)

b Show that AB $\perp$ BC. (3)

c Prove that ABCD is a cyclic quadrilateral. (4)

d Determine the equation of the circle ABCD. (7)

[18]

6 Given the vertices $A(2;3)$, $B(5;4)$, $C(4;2)$ and $D(1;1)$ of parallelogram ABCD. Determine:

a the coordinates of M, the point of intersection of diagonals AC and BD (2)

b the equation of the median PM of ΔDMC (5)

[7]

7 a If $secA = \frac{5}{4}$ and $180° < A < 360°$, determine the following without the use of a calculator:

i $cosA$ (1)

ii $sin2A$ (4)

b If $sin17° = k$, express $\frac{cosec73°}{cos343°}$ in terms of k. (3)

[8]

8 a Determine the value of the following without using a calculator:

$cos69°.cos9° + cos81°.cos21°$ (4)

b Consider the following identity:

$$\frac{1+cosx+cos2x}{sinx+sin2x} = \frac{1}{tanx}$$

i For which values of x will the identity be undefined? (4)

i Prove the identity. (4)

[12]

9 a Solve the following equations for the interval $[-90°; 90°]$:

i $2tanx = -0{,}6842$ (2)

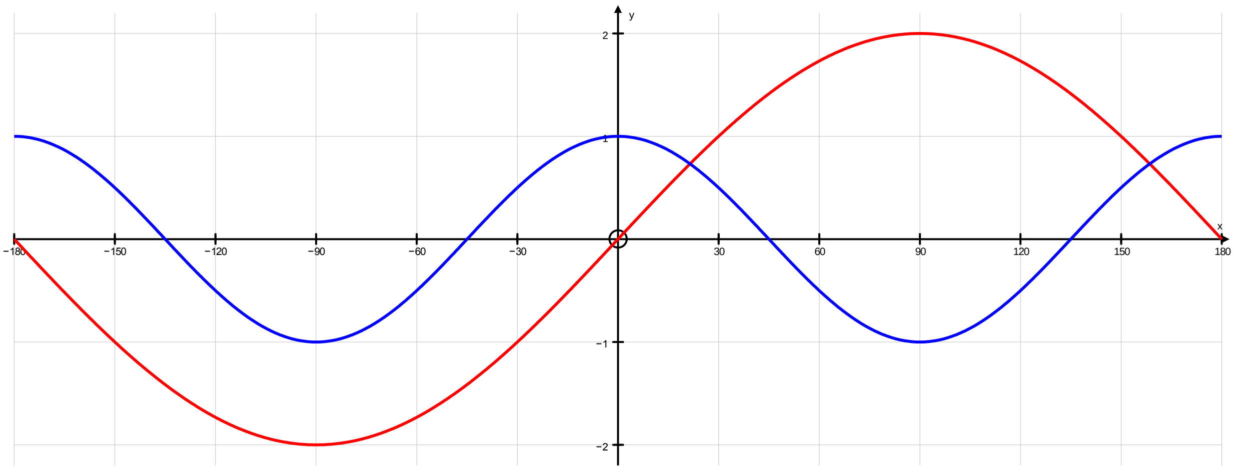
ii $sin2x.cosx - sinx.cos2x = 0{,}5$ (2)

b Determine the general solution of:

$\cos\left(\frac{1}{2}x + 15°\right) = \sin(2x - 15°)$ (5)

[9]

10 The graphs of $y = asinx$ and $y = cosbx$ are drawn over the interval

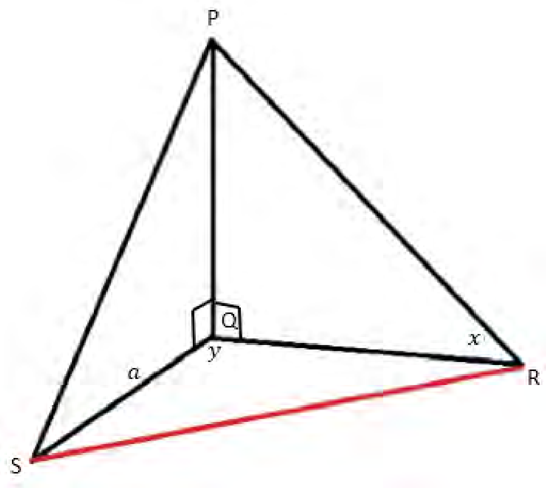


a Write down the values of a and b. (2)

b Use your graph to determine approximate values of x; $x \in [-180°; 180]$ for which $cos^2x - sinx = \frac{1}{2}$ (5)

[7]

11 In the diagram below, Q is the base of a vertical tower PQ, while R and S are points in the same horizontal plane as Q. The angle of elevation of P, the top of the tower, as measured from R, is x. Furthermore, $R\hat{Q}S = y$, $QS = a\ metres$ and the area of $\Delta QRS = A\ m^2$.



a Show that $PQ = \frac{2Atanx}{asiny}$ (5)

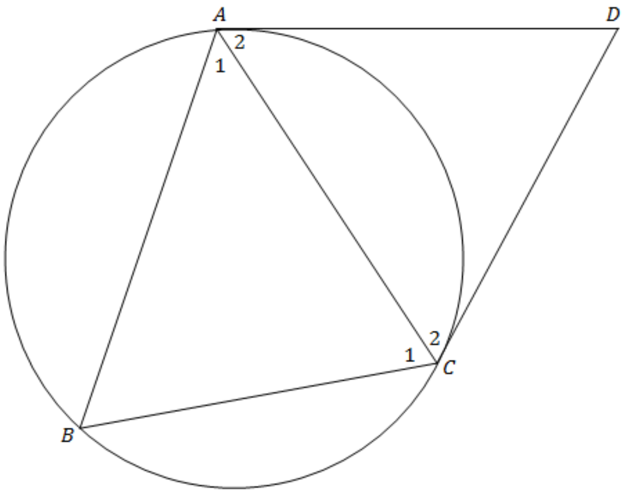
b Calculate the value of y if $PQ = 76{,}8m$; $a = 87{,}36$; $A = 480{,}9m^2$ and $x = 46{,}5°$. (3)

[8]

12 a Write down the converse of the following theorem:

The angle between a tangent to a circle and a chord drawn through the point of contact, is equal to an angle in the alternate segment. (2)

b The diagonal AC of quadrilateral ABCD bisects $B\hat{C}D$ while AD is a tangent to the circle ABC at point A. Prove that AB is a tangent to circle ACD. (5)

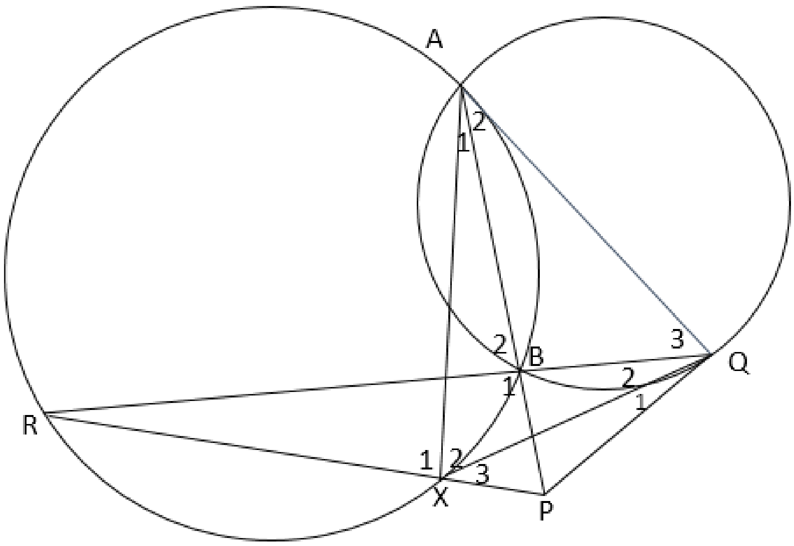


c Two circles intersect at A and B. AB is produced to P. PQ is a tangent to the smaller circle at Q. QB produced meets the larger circle at R. PR cuts the larger circle at X. AX and AQ are drawn.

Prove that:

i Points A, X, P and Q are on the circumference of the same circle. (5)

ii PQ is a tangent to the circumscribed circle of ΔQRX. (3)



[15]

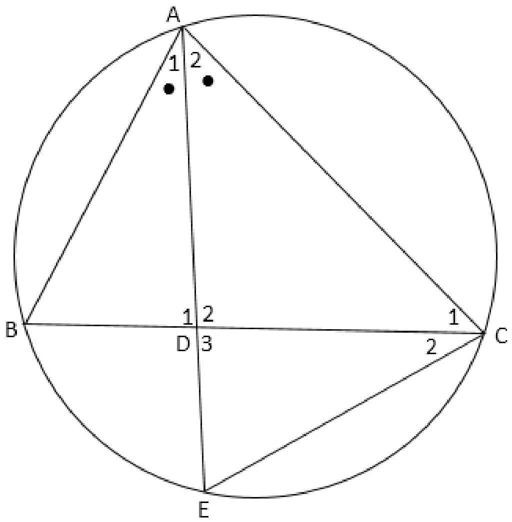
13 a In ΔABC and ΔDEF, $\hat{A} = \widehat{D}$ and $\hat{B} = \hat{E}$.

Prove the theorem that $\frac{DE}{AB} = \frac{EF}{BC} = \frac{DF}{AC}$. (8)

b In the diagram A, B, C and E are points on a circle.

AE bisects $B\hat{A}C$ and BC.

AE intersect in D.



Prove that:

i $\Delta ABD /// \Delta AEC$ (4)

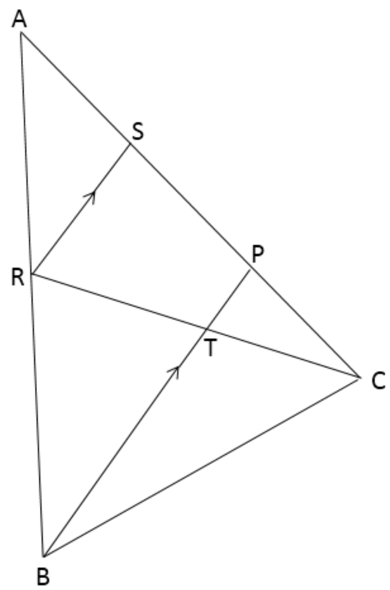
ii $AB.AC = AD^2 + BD.DC$ (7)

[19]

14 In ΔABC , P is the midpoint of AC, RS//BP and $\frac{AR}{AB} = \frac{3}{5}$.

CR and BP intersect at T.

Determine, giving reasons, the following ratios:



a $\frac{AS}{SP}$ (4)

b $\frac{AS}{SC}$ (3)

c $\frac{RT}{TC}$ (3)

d $\frac{Area\ \Delta TPC}{Area\ \Delta RSC}$ (6)

[15]

MEMORANDUM: Exemplar Paper 2

1 a Fourth quarter. Spread $= 13 - 12 = 1$

b IQR$= 12 - 2 = 10$

c More data in 10-13

Median = 10 and Max=13. Therefore 50% of the data lies in interval 10-13

25% of data lies between 2-10. Therefore less than 50% in 5-10

d 2-4 has fewest data

0-2, 2-10 ,10-12 and 12-13 all represent 25% of the data

2-4 will only be a part of 25% (less than 25%)

2 a $y = 767{,}867 + 98{,}514x$

b $3254 = 767{,}867 + 98{,}514x$

$98{,}514x = 2486{,}133$

$x = 25{,}236$

Shipping will be done on the 26th day.

3 a 100

b 80-20=60

c 14 minutes

4 a Midpoint$= \left(\frac{1+7}{2}; \frac{4+2}{2}\right) = (4; 3)$

Radius $= \sqrt{(4-1)^2 + (3-4)^2} = \sqrt{10}$

$(x-4)^2 + (y-3)^2 = 10$

b KB $\perp$ BC ($\hat{B} = 90°$; angle in semi circle)

$m_{BC} = -2$

$\therefore m_{KB} = \frac{y-4}{x-1} = \frac{1}{2}$ $\qquad 2(y-4) = x - 1$ .....(1)

$\therefore m_{BC} = \frac{y-2}{x-7} = -2$ $\qquad (y-2) = -2(x-7)$ ... ... (2)

Solving equations (1) and (2) simultaneously yields:

$x = 5\,; y = 6$

5 a $m_{BC} = \frac{3-2}{1-4} = -\frac{1}{3}$ $\quad m_{CD} = \frac{4-2}{8-4} = \frac{1}{2}$

b $m_{AB} = \frac{12-3}{4-1} = 3$ $\quad \therefore m_{AB} \times m_{BC} = -\frac{1}{3} \times 3 = -1$

$\therefore \mathrm{AB} \perp \mathrm{BC}$

c $m_{AD} = \frac{12-4}{4-8} = -2$ $\quad \therefore m_{CD} \times m_{AD} = \frac{1}{2} \times -2 = -1$

$\therefore \mathrm{AD} \perp \mathrm{CD}$

$\hat{B} = 90°$ from 5 b $\quad \hat{D} = 90°$

$\hat{B} + \hat{D} = 180°$

ABCD is a cyclic quad (opp angles supp)

d AC is diameter of circle(angles in semi circle =90°)

Midpoint of AC= $\left(\frac{4+4}{2}; \frac{12+2}{2}\right) = (4; 7)$

Radius $= 12 - 7 = 5$

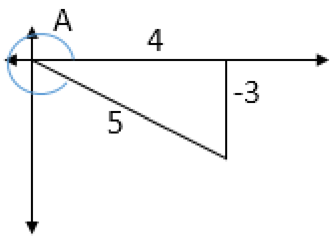
$(x-4)^2 + (y-7)^2 = 25$

6 a M$\left(\frac{2+4}{2}; \frac{3+2}{2}\right) = \left(3; \frac{5}{2}\right)$ (diagonals bisect each other)

b Median PM join M with point P on DC, where P is the midpoint of DC

P$\left(\frac{1+4}{2}; \frac{1+2}{2}\right) = \left(\frac{5}{2}; \frac{3}{2}\right)$

7 a i $cosA = \frac{1}{secA} = \frac{4}{5}$

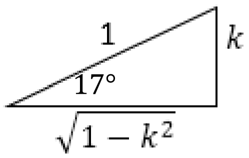


ii $sin2A = 2sinAcosA$

$= 2 \times \frac{-3}{5} \times \frac{4}{5}$

$= \frac{-24}{25}$

b



$\frac{cosec73°}{cos343°} = \frac{cosec17°}{\cos 17°} = \frac{\frac{1}{k}}{\frac{\sqrt{k^2-1}}{1}} = \frac{1}{k\sqrt{k^2-1}}$

8 a $cos69°.cos9° + cos81°.cos21° = sin21°.cos9° + sin9°.cos21°$

$= \sin(21° + 9°)$

$= \sin 30° = \frac{1}{2}$

b i Undefined at asymptotes of $\tan x$: $x = 90° + k.180°; k \in Z$

Also undefined where $sinx + sin2x = 0$

$sinx + 2sinxcosx = 0$

$sinx(1 + 2cosx) = 0$

$\therefore sinx = 0$ or $cosx = -\frac{1}{2}$

$x = k.360°$ or $x = \pm 120° + k.360°; k \in Z$

ii $\frac{1+cosx+cos2x}{sinx+sin2x} = \frac{1+cosx+2cos^2x-1}{sinx+2sinxcos}$

$= \frac{cosx(1+2cosx)}{sinx(1+2cosx)}$

$= \frac{cosx}{sinx}$

$= \frac{1}{tanx}$

9 a i $2tanx = -0,6842$

$tanx = -0,3421$

$x = -18,89°$

ii $sin2x.cosx - sinx.cos2x = 0,5$

$\sin(2x - x) = 0,5$

$sinx = 0,5$

$x = 60°$

b $\cos\left(\frac{1}{2}x + 15°\right) = \sin(2x - 15°)$

$\cos\left(\frac{1}{2}x + 15°\right) = \cos[90° - (2x - 15°)]$

$\cos\left(\frac{1}{2}x + 15°\right) = \cos[90° - (2x - 15°)]$

$\cos\left(\frac{1}{2}x + 15°\right) = \cos(105° - 2x)$

$\left(\frac{1}{2}x + 15°\right) = (105° - 2x) + k.360°$

or $\left(\frac{1}{2}x + 15°\right) = -(105° - 2x) + k.360°$

$x = 36° + k.144°; k \in Z$ $x = 80° - k.240°; k \in Z$

10 a $a = 2; b = 2$

b $cos^2x - sinx = \frac{1}{2}$

$2cos^2x - 2sinx = 1$

$2cos^2x - 1 = 2sinx$

$\therefore$ It is where the two graphs meet.

$x = 20°\ or\ 160°$

11 a $tanx = \frac{PQ}{QR}$ $\therefore PQ = QRtanx$

Area of $\Delta QRS = \frac{1}{2}QS.QRsinQ\hat{R}S$

$\therefore A = \frac{1}{2}a \times QRsiny$

$QR = \frac{2A}{asiny}$

$PQ = \frac{2A}{asiny}tanx = \frac{2Atanx}{asiny}$

b $76,8 = \frac{2(480,9)tan46,5°}{87,36siny}$

$siny = \frac{2(480,9)tan46,5°}{87,36(76,8)} = 0,151064$

$y = 8,69°\ or\ 171,31°$

12 a If a line is drawn through the endpoint of a chord to form an angle which is equal to the angle in the opposite segment, then this line is a tangent.

b $\hat{A}_2 = \hat{B} = x$ tan chord

$\hat{C}_1 = \hat{C}_2 = y$ given

$\hat{A}_1 = 180° - (x + y)$ sum of angles of ΔABC

$\hat{D} = 180° - (x + y)$ sum of angles of ΔADC

$\therefore \hat{A}_1 = \hat{D}$

$\therefore$ AB is a tangent to the circle

c i $\hat{X}_1 = \hat{B}_2$ angles in same segm

$= \hat{A}_2 + \hat{Q}_3$ ext angle of triangle

But

$\hat{A}_2 = \hat{Q}_1 + \hat{Q}_2$ tan chord

$\therefore \hat{X}_1 = \hat{Q}_1 + \hat{Q}_2 + \hat{Q}_3$

$\therefore$ A,X,P,Q concyclic (ext angle = opp int angle)

ii $\hat{Q}_1 = \hat{A}_1$ AXPQ cyclic quad

$= \hat{R}$ angles in same segm

$\therefore$ PQ is a tangent

13 a Book work

b i In ΔABD and ΔAEC:

$\hat{A}_1 = \hat{A}_2$ given

$\hat{B} = \hat{E}$ angles in same segm

∴ΔABD|||ΔAEC (AAA)

ii In ΔABD and ΔCED:

$\hat{B} = \hat{E}$ proven

$\hat{D}_1 = \hat{D}_2$ vert opp ∠s

∴ΔABD|||ΔCED (AAA)

$\therefore \frac{AB}{AE} = \frac{AD}{AC}$

$\therefore AB.AC = AE.AD$

$= (AD + DE)AD$

$= AD^2 + AD.DE$

But AD.DE=BD.DC $\left(\frac{AD}{DC} = \frac{BD}{DE}\right)$

$\therefore AB.BC = AD^2 + BD.DC$

14 a $\frac{AR}{AB} = \frac{3}{5}$ given

Let $AR = 3k$ and $AB = 5k$

$\therefore \frac{AS}{SP} = \frac{3}{2}$ RS//BP

b Let $AS = 3m$ and $AP = 5m$

but AP=PC (given)

$\therefore AP = PC = 5m$

$\therefore \frac{AS}{SC} = \frac{3m}{7m} = \frac{3}{7}$

c $\frac{RT}{TC} = \frac{2m}{5m}$ RS//TP

$= \frac{2}{5}$

d $\frac{Area\Delta TPC}{Area\Delta RSC} = \frac{\frac{1}{2}TC.PC.sin\hat{ACR}}{\frac{1}{2}RC.SC.sin\hat{ACR}}$

$= \frac{TC}{RC}.\frac{PC}{SC} = \frac{5}{7}.\frac{5}{7} = \frac{25}{49}$