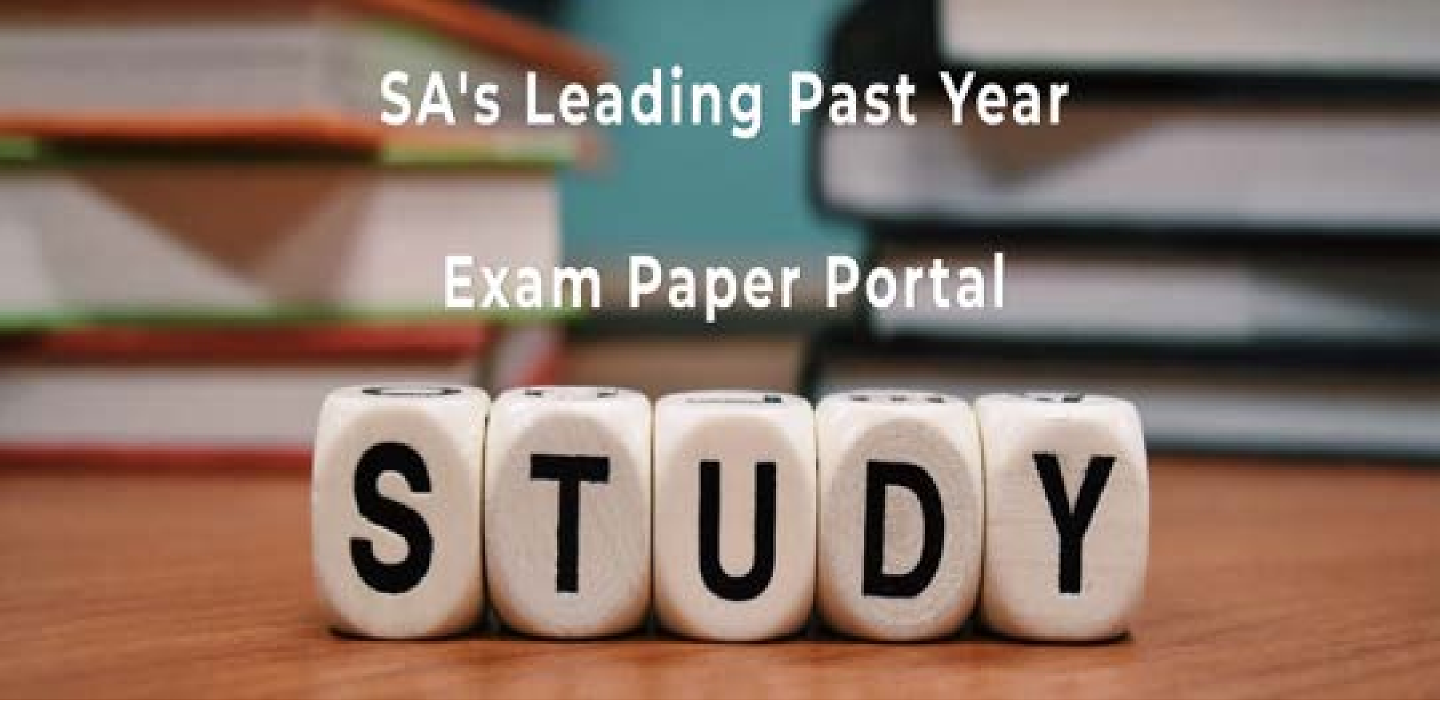
SA's Leading Past Year
Exam Paper Portal
STUDY

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1. Introduction

The declaration of COVID-19 as a global pandemic by the World Health Organisation led to the disruption of effective teaching and learning in many schools in South Africa. The majority of learners in various grades spent less time in class due to the phased-in approach and rotational/ alternate attendance system that was implemented by various provinces. Consequently, the majority of schools were not able to complete all the relevant content designed for specific grades in accordance with the Curriculum and Assessment Policy Statements in most subjects.

As part of mitigating against the impact of COVID-19 on the current Grade 12, the Department of Basic Education (DBE) worked in collaboration with subject specialists from various Provincial Education Departments (PEDs) developed this Self-Study Guide. The Study Guide covers those topics, skills and concepts that are located in Grade 12, that are critical to lay the foundation for Grade 12. The main aim is to close the pre-existing content gaps in order to strengthen the mastery of subject knowledge in Grade 12. More importantly, the Study Guide will engender the attitudes in the learners to learning independently while mastering the core cross-cutting concepts.

2. How to use this Self Study Guide?

- This Self Study Guide summaries only two topics, Functions and Finance. Hence the prescribed textbooks must be used to find more exercises.
- It highlights key concepts which must be known by all learners.
- Deeper insight into the relevance of each of the formulae and under which circumstances it can be used is very essential.
- Learners should know the variables in each formula and its role in the formula. Learners should distinguish variable in different formulae.
- More practice in each topic is very essential for you to understand mathematical concepts.
- The learners must read the question very carefully and make sure that they understand what is asked and then answer the question.
- After answering all questions in this Self Study Guide, try to answer the previous question paper to gauge your understanding of the concepts your required to know.

3. Functions

3.1 Notes/Summaries/Key Concepts

What learners must know:

- Revise the functions discussed in Grade 10:

  Straight line: $y = ax + q$

  Parabola: $y = ax^2 + q$

  Hyperbola: $y = \frac{a}{x} + q$

  Exponential graphs: $y = a.b^x + q$

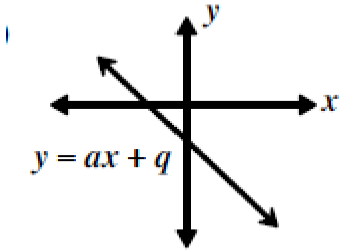
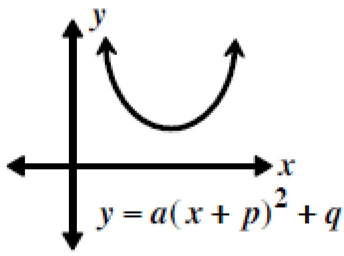
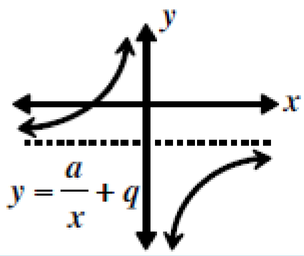
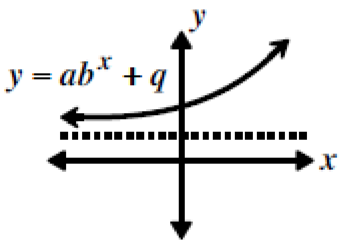
- Effects of *a*, *p* and *q* on the graphs of the functions:

  Parabola: $y = a(x+p)^2 + p$

  Hyperbola: $y = \frac{a}{x+p} + q$

  Exponential graphs: $y = a.b^{x+p} + q$ where $b > 0,\ b \neq 1$

- to differentiate between different shapes of graphs based on the equation.
- to draw graphs
- to analyze and interpret graphs, make deductions from the given graphs.
- how to recognize the domain and the range
- identify the asymptote (s).
- how to find the point (s) of intersection
- that hyperbola must have arms at all times, and show at least one point on the other arm
- to investigate the average gradient between two points on a curve

TYPES OF FUNCTION		
Function	**Equation**	**Shape**
1. Linear Function	$f(x) = mx + c$ or $f(x) = ax + q$	
2. Quadratics Function	i) $f(x) = ax^2$ ii) $f(x) = ax^2 + q$ iii) $f(x) = ax^2 + bx + c$ iv) $f(x) = a(x+p)^2 + q$ v) $f(x) = a(x - x_1)(x - x_2)$	
3. Hyperbolic Function	• $y = \frac{a}{x} + q$ • $y = \frac{a}{x+p} + q$	
4. Exponential Function	i) $f(x) = ab^x + q$ ii) $f(x) = ab^{x+p} + q$	

Linear Function

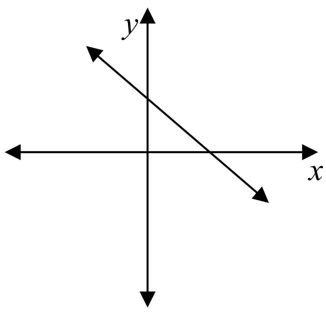
a) Standard form is given by $f(x) = mx + c$ or $f(x) = ax + q$

a and m represent the gradient or slope

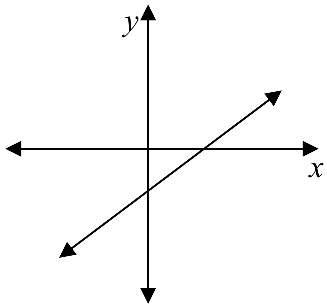
c and q are the y-intercept

b) Shape of the graph

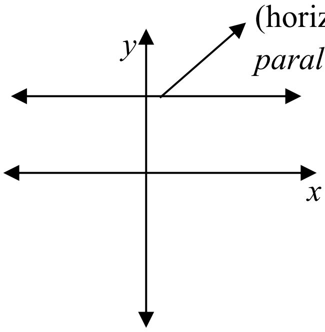
$m < 0$ and $a < 0$



$m > 0$ and $a > 0$

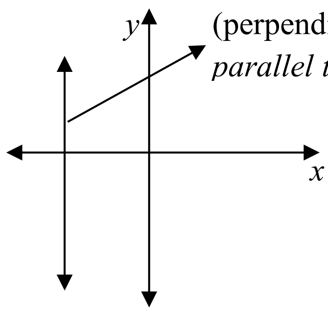


$m = 0$ and $a = 0$



(horizontal line)
parallel to x-axis

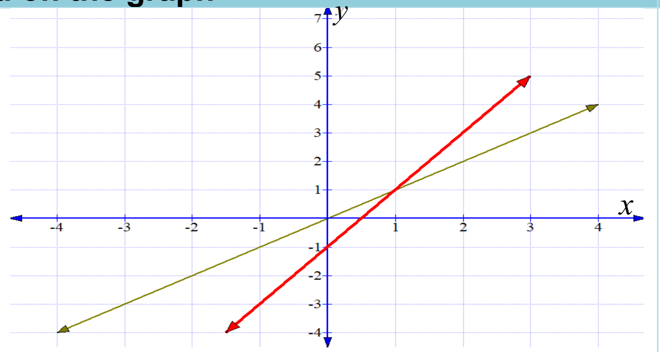
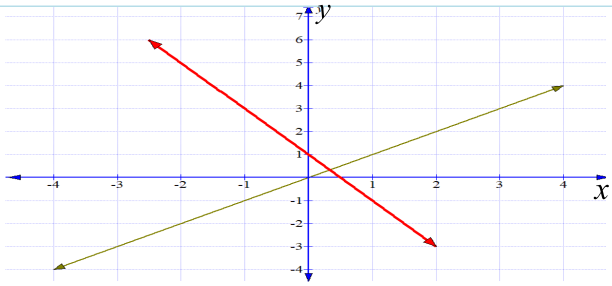
m and a undefined

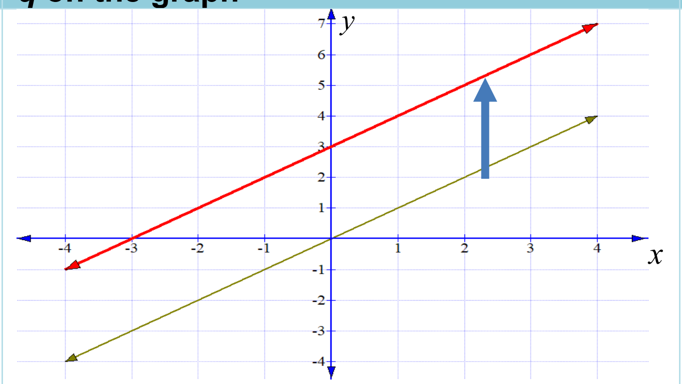
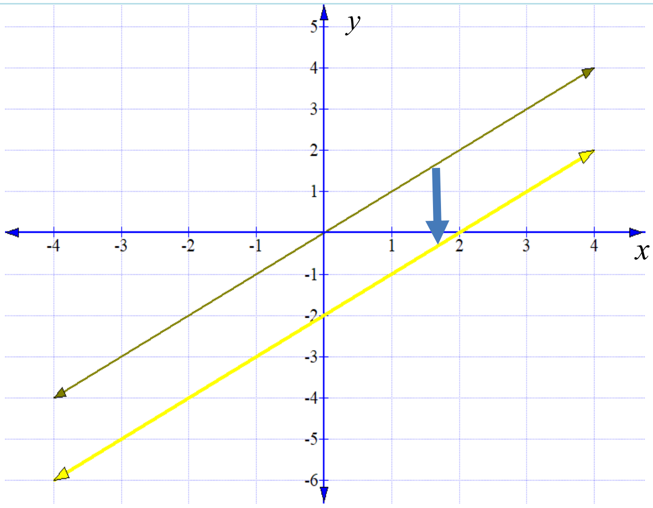


(perpendicular line)
parallel to y-axis

c) Domain: $x \in \mathbb{R}$
Range: $y \in \mathbb{R}$

Effect of *a* and *q* on the graph

Function	*a* on the graph	Notes
$f(x)=2x$		If $a>0$, the line slopes up from left to right. As the positive value of *a* increases, the straight line gets closer to the *y*-axis.
$f(x)=-2x$		If $a<0$, the line slopes down from left to right. As the negative value of a decreases, the straight line gets closed to the y-axis.

Function	*q* on the graph	Notes
$f(x)=x+3$		$q>0$ graph move or shift *q* units up and the *y*-intercept is on the positive side.
$f(x)=x-2$		$q<0$, graph move or shift *q* units downwards and *y*-intercept is negative side

Sketching the graph

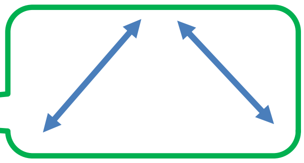
- The following steps must be taken into account when sketching

Let $y = 0$

Let $x = 0$

Steps:

- Find *x*- and *y*- intercepts or
- Use table method
- Determine the shape



Worked-out example

Given: $f(x) = 2x - 4$

a) Calculate x - and y - intercept of *f.* (4)

b) Sketch the graph of *f.* (2)

c) Determine the range of *f.*

(1)

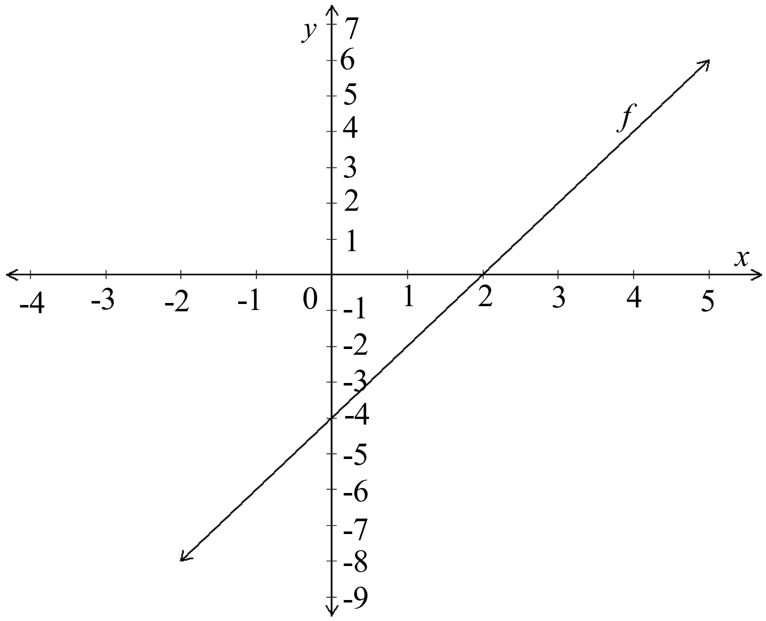
d) Determine the gradient of $h(x)$, if $h(x) = -f(x)$ (2)

e) Calculate the coordinates of point of intersection of $f(x)$ and $g(x)$

(2)

Possible solution

a)	y -intercept: let $x = 0$ $y = 2(0) - 4$ $= -4 \quad (0\ ;\ \text{-}4)$ x -intercept: let $y = 0$ $0 = 2x - 4$ $4 = 2x$ $\therefore\ x = 2\ (2\ ;\ 0)$	✓ subst. $x = 0$ ✓ answer ✓ subst. $y = 0$ ✓ answer (4)

b)		✓ both *x*- & *y*-inter ✓ correct shape (2)
c)	$y \in {}^{\sim}$ values of y form part of the graph.	✓ answer (1)
d)	$h(x) = -(2x-4)$ Subst. equation of f. $= -2x+4$ $\therefore$ Gradient is -2 or $m = -2$	✓ equation ✓ $m = -2$ (2)
	$2x-4 = -2x+4$ $2x+2x = 4+4$ $4x = 8$ $\frac{4x}{4} = \frac{8}{2}$ $\therefore x = 2$	✓ equating *h* & *f* ✓ answer (2)

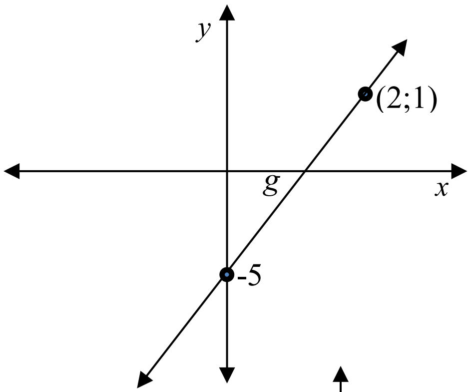
Finding the equation of straight line (linear function)

- Need **gradient** and **one point** on the line
- **NB**: *m* can be found in different way
  - Given two coordinates, use $m = \dfrac{y_2 - y_1}{x_2 - x_1}$
  - Parallel functions, $m_1 = m_2$
  - Perpendicular functions, $m_1 \times m_2 = -1$

Worked-out example

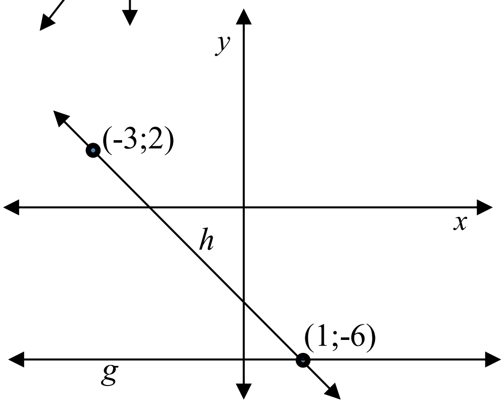
1. In the diagram below, determine the equation *g, f* and *h* in the form $g(x) = ax + q$.

i)



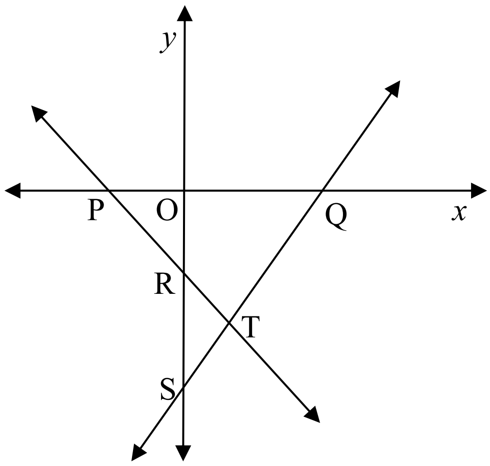
(3)

ii)



(3)

2. In the diagram, line $x - 2y = 3$ cuts the axis at Q and S. Line $x + y = -1$ cuts the axes at P and R. The two lines intersect at T.



Determine:

a) the coordinates of Q and S. (4)

b) the coordinates of P. (1)

c) coordinates of R. (1)

d) the value(s) of *x* for which the lines are increasing. (1)

[13]

Possible solutions

1. **i)**	$m = \dfrac{y_2 - y_1}{x_2 - x_1}$ Subst. point Q & S into gradient formula $= \dfrac{1-(-5)}{2-0}$ $= \dfrac{6}{2} = 3$ $\therefore y = ax + q$ $1 = 3(2) + c$ or $y = 3x - 5$ $y = 3x - 5$	✓ correct subst. ✓ answer ✓ equation (3)
ii)	$m = \dfrac{y_2 - y_1}{x_2 - x_1}$ Subst. point P & R into gradient formula $= \dfrac{2-6}{-3-1}$ $= -2$ $y = -2x + q$ $2 = -2(-3) + q$ $q = -4$ $y = -2x - 4$	✓ correct subst. ✓ answer ✓ equation (3)
2. a)	$0 = \dfrac{1}{2}x - 3$ Q is on the x-axis, x-intercept, let $y = 0$ $3 = \dfrac{1}{2}x$ $x = 6$ Coordinates of S (6;0) $: \ y = \dfrac{1}{2}(0) - 3$ S is on the x-axis, y-intercept, let $y = 0$ $= -3$ Coordinates of S(0;-3)	✓ equating y to 0 ✓ answer ✓ equating x to 0 ✓ answer (4)
b)	Coordinates of P(-1;0)	✓ P(-1;0) (4)
c)	Coordinates of R(0;-1)	✓ R(0;-1) (2)
d)	Increase when $x \in \sim$	✓ answer (1)

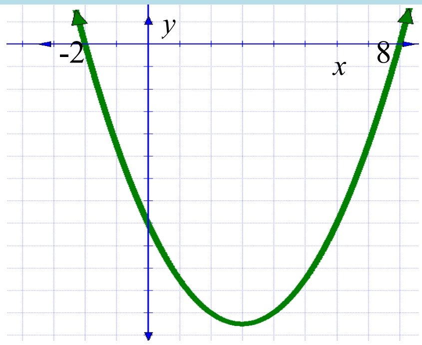
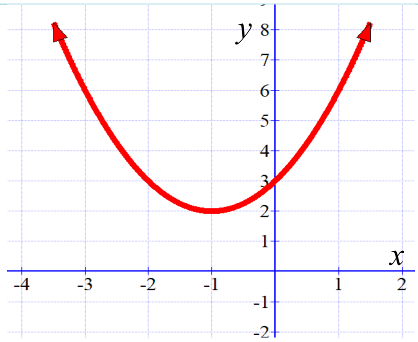
Quadratic function

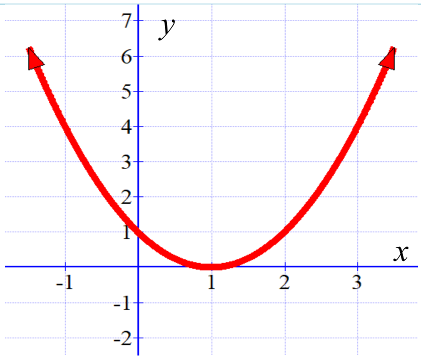
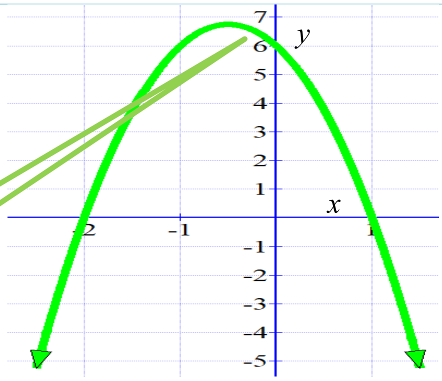
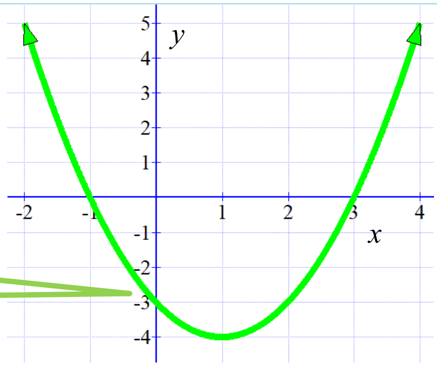
Standard form is given by: $y = ax^2 + q$

$$y = ax^2 + bx + c$$

$$y = a(x+p)^2 + q$$

SHAPE	
Happy face parabola (**Concave up**)	Sad face parabola (**Concave down**)
$a > 0$  Min TP $y \in [q;\infty)$	$a < 0$  Max TP $y \in (\infty; q]$
The graph has a **minimum** turning point	The graph has a **maximum** turning point

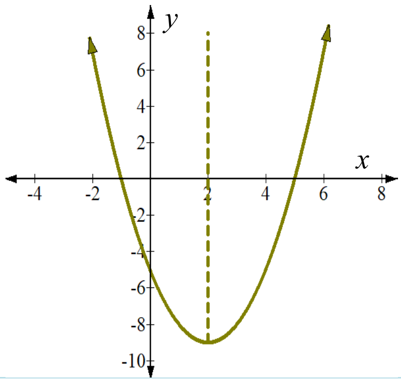
INTERCEPTS WITH AXIS			
Interprets	**Example**	**graph**	**Nature of root**
***x*-intercepts** Let $y = 0$	$f(x) = x^2 - 6x - 16$ **Example** $0 = x^2 - 6x - 16$ $0 = (x+2)(x-8)$ $x = -2$ or $x = 8$		$\Delta > 0$ and **Two real,** unequal **and** **rational roots**
	$f(x) = x^2 + 2x + 3$ **Example** $0 = x^2 + 2x + 3$ $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $= \dfrac{-2 \pm \sqrt{(2)^2 - 4(1)(3)}}{2(1)}$ $= \dfrac{-2 \pm \sqrt{-10}}{2}$	 No real roots	$\Delta < 0$ **Non real roots**

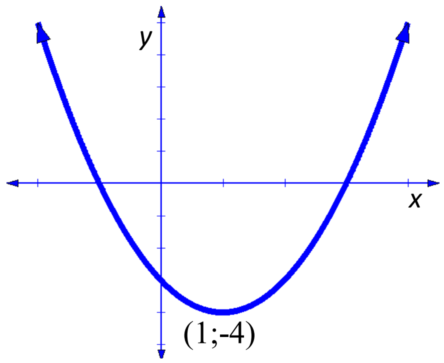
	$f(x)=x^2-2x+1$ **Example** $0=x^2-2x+1$ $0=(x-1)(x-1)$ $\therefore x=1$		$\Delta > 0$, and **Two real,** equal, **and** **rational roots**
y-intercept Let $x=0$	$g(x)=3x^2-4x+1$ **Example** $y=3(0)^2-4(0)+6$ $=6$ y - intercept		If the value of ***c*** **is positive**, graph will cut the ***y*-axis on the positive** side
	$g(x)=x^2-2x-3$ **Example** $y=(0)^2-2(0)-3$ $y=-3$ y - intercept		If the value of ***c*** **is negative**, graph will cut the ***y*-axis on the negative** side

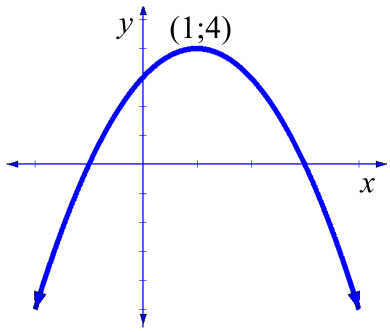
AXIS OF SYMMETRY

Axis of symmetry divides the graph into two qual halves

Equation	**Use**	**Example**
$f(x)=ax^2+bx+c$	$x=-\frac{b}{2a}$	$f(x)=x^2-4x-5$ $x=-\frac{(-4)}{2(1)}=-\frac{4}{2}=-2$
$f(x)=a(x+p)^2+q$	$x+p=0$ $x=-p$	$g(x)=2(x+5)^2-9$ $x+5=0$ $x=-5$

When given $f(x)=(x-x_1)(x-x_2)$ or roots or *x*-intercepts of quadratic function *f.*	$x=\dfrac{x_1+x_2}{2}$	$f(x)=(x-1)(x+5)$ $x=\dfrac{1+(-5)}{2}=\dfrac{1-5}{2}==-2$ or 

DOMAIN AND RANGE OF QUADRATIC FUNCTION	
Domain	$x\in\mathbb{R}$ This can also be written as $x\in(-\infty;\infty)$
Range	$y\in\mathbb{R}$, $y\geq q$; *if* $a>0$ This can also be written as $y\in[q;\infty)$. $\text{Range}=\{y/\,y\geq \text{minimum value}\}$ **Example**  **Minimum Turning Point** Range: $y\geq -4$ or $y\in[-4;\infty)$ or
	$y\in\mathbb{R}$, $y\leq q$; *if* $a<0$ This can also be written as $y\in(-\infty;q]$ $\text{Range}=\{y/\,y\leq \text{maximum value}\}$

	 **Maximum Turning Point** Range: $y \in \mathbb{R}$, $y \leq 4$ or $y \in \left(-\infty; 4\right]$

TURNING POINT		
Standard equation	*x*-coordinate of turning point (Axis of symmetry)	*y*-coordinate of the turning point (*Maximum* or *minimum value*)
$y = ax^2 + bx + c$	$x = -\dfrac{b}{2a}$	Substitute the ***x*-value** into the original equation to determine the *y*-value
$y = a(x+p)^2 + q$	$x = -p$	$y = q$
$y = a(x - x_1)(x - x_2)$	$x = \dfrac{x_1 + x_2}{2}$	Substitute the *x*-value into the original equation to determine the *y*-value
$f(x) = ax^2 + bx + c$	$x = f'x)$, where $f'x) = 0$	Substitute the *x*-value into the original equation to determine the *y*-value

Sketching a graph

- Determine the shape of *a*, where $a > 0$ or $a < 0$.
- Find *x*-intercepts by letting $y = 0$
- Find *y*-intercept by letting $x = 0$
- Find turning point:
- Plot points and sketch graph

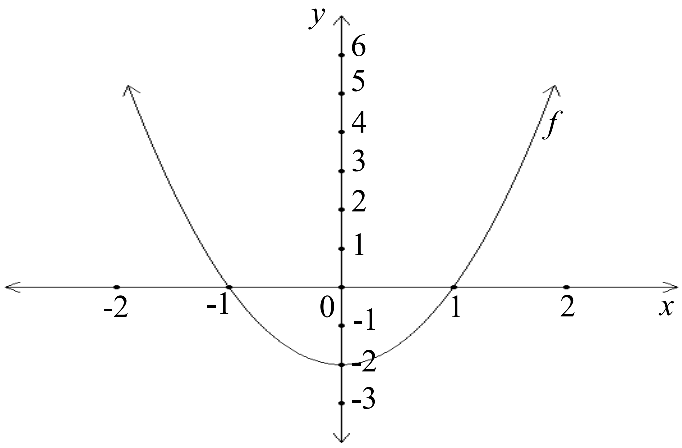
Worked-out example 1

Given: $f(x)=2x^2-2$.

1.1	Determine the coordinates of the *y*- intercept of the graph of *f*.	(2)
1.2	Determine the coordinates of the *x*-intercepts of *f*.	(3)
1.3	Determine the coordinates of the turning point of *f*.	(2)
1.4	Sketch the graphs of *f*, clearly showing ALL intercepts with the axes and turning points.	(3)
1.5	Write down the range of *f*.	(1)
1.6	Determine a if $(2;a)$ is a point on the graph of $f(x)$.	(2)

[13]

Possible solutions

1.1	*y* – intercept, let *x* = 0 $y=2(0)^2-2$ $y=2 \quad (0;2)$	✓ *x* = 0 ✓ answer (2)
1.2	*x* – intercepts, let *y* = 0 $0=2x^2-2$ $0=x^2-1$ $0=(x-1)(x+1)$ $\therefore x=1 \quad \text{or} \quad x=-1$ $(1;0) \qquad (-1;0)$	✓ *y* = 0 ✓ factors ✓ both *x* values (3)
1.3	Turning Point: since $a>0$ $(-2;0)$	✓✓ $(-2;0)$ (2)
1.4		✓ intercepts ✓ turning point ✓ shape (4)
1.5	Range: $y\in[-2;\infty)$	✓ critical values ✓ notation (2)

Worked-out example 2

Given: $f(x)=x^2-4x-12$

2.1 Write down the coordinate of *y*-intercept of *f*. (1)

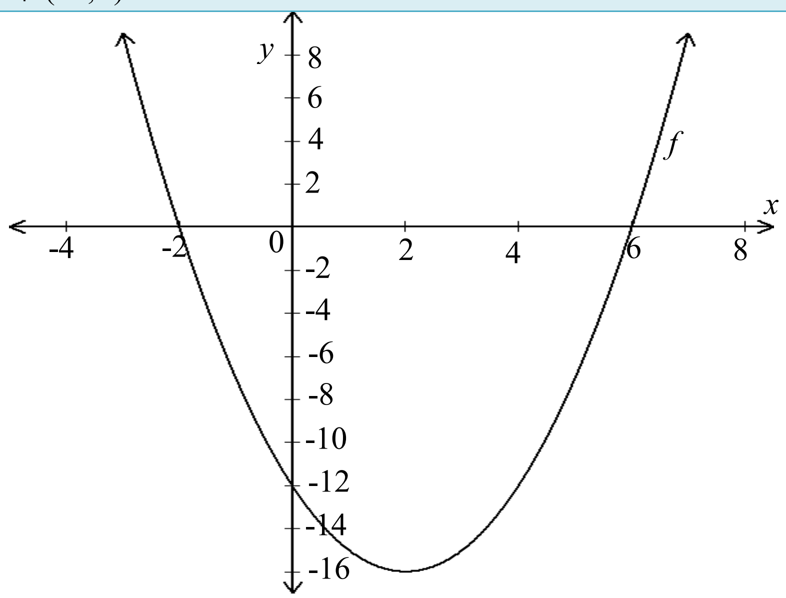
2.2 Determine *x*-intercepts of *f*. (3)

2.3 Determine the coordinates of turning point of *f*. (3)

2.4 Sketch the graph of *f*, clearly show all intercepts with axis. (3)

2.5 Determine the range of *f*. (2)

Possible solutions

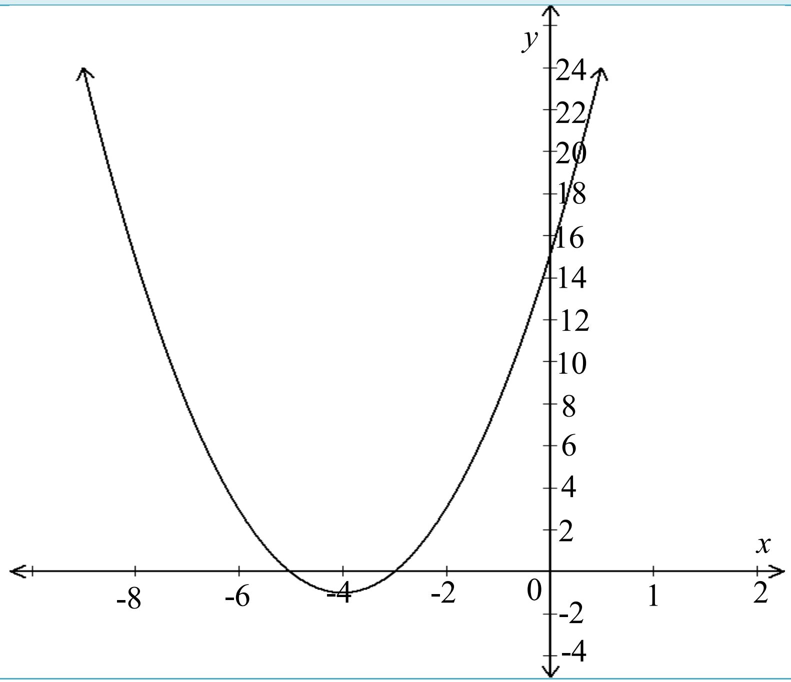
2.1	$(0;-12)$	✓ $(0;-12)$ (1)
2.2	$0=x^2-4x-12$ $0=(x-6)(x+2)$ Find factors $x-6=0$ or $x+2=0$ $\therefore x=6$ or $x=-2$	✓ equating *y* to 0 ✓ factors ✓ both answer (3)
2.3	$x=-\dfrac{b}{2a}$ $x=-\dfrac{-6}{2(1)}$ $=-3$ $y=(-3)^2-4(-3)-12$ $=9$ Tp $(-3;9)$	✓ Subst. *a* & *b* ✓ value of *x* ✓ value of *y* (3)
2.4		✓ *x*- & *y*-intercepts ✓ turning point ✓ shape (3)
2.5	$y\in[-16;\infty)$	✓ answer ✓ notation (2)
		[12]

Worked-out example 3

Consider graph of $g(x)=(x+4)^2-1$

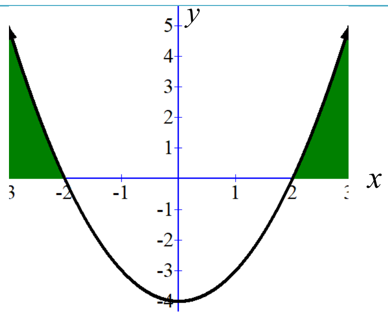
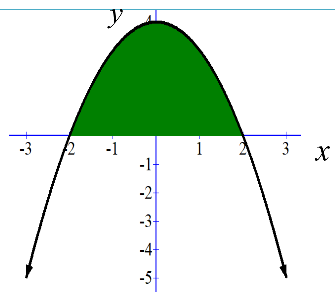
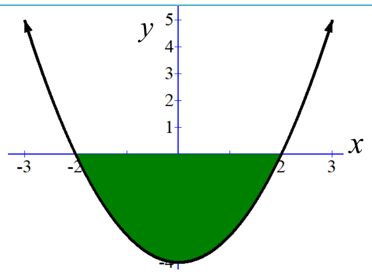
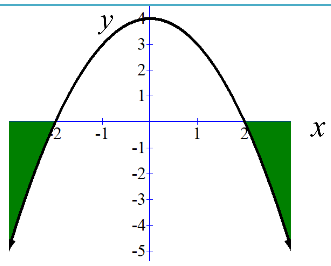
3.1	Write down the coordinates of turning point of *g*.	(2)
3.2	Determine *x*-intercepts of *g*.	(3)
3.3	Write down the range of *g*.	(2)
3.4	Sketch the graph of *g*.	(3)
		[10]

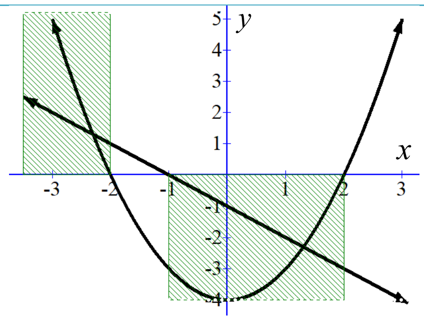
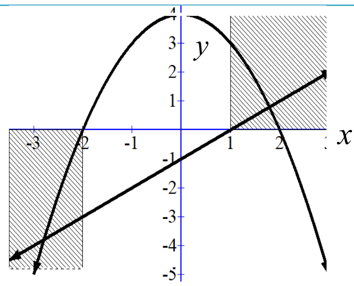
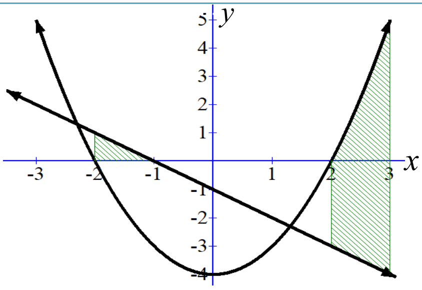
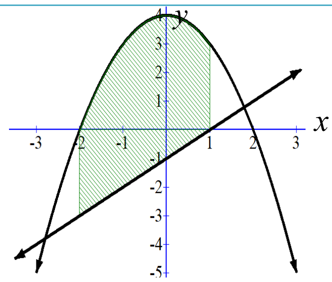
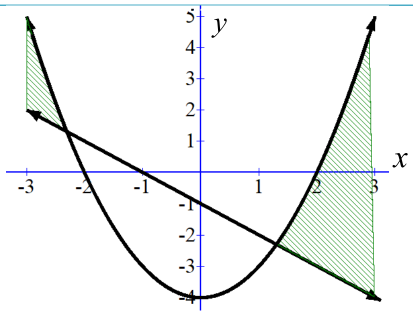
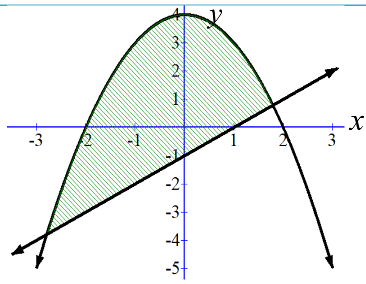
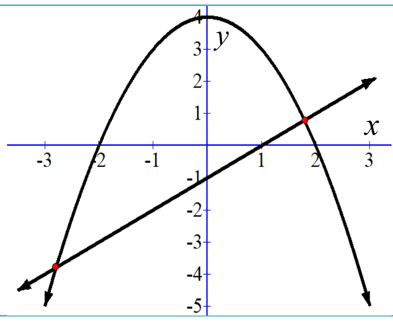
Possible solution

3.1	TP: $(-4;-1)$	✓ -4 ✓ -1 (2)
3.2	$0=(x+4)^2-1$ $1=(x+4)^2$ $\pm1=x+4$ $\therefore x=1-4$ or $x=-1-4$ $=-3$ $\quad$ $=-5$	✓ equating *y* to 0 ✓ factors ✓ answers (3)
3.3	$y\in[-1;\infty)$	✓ critical values ✓ notation (2)
3.4		✓ *x*- & *y*-intercepts ✓ turning point ✓ shape (3)

DETERMINING THE EQUATION OF A PARABOLIC FUNCTION

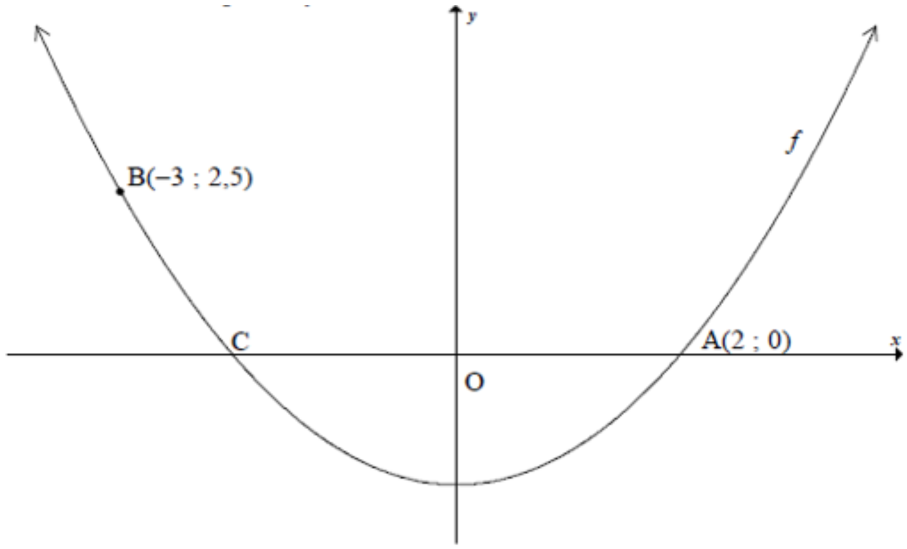
Given x-intercepts (roots) and a point on the graph	**Given Turning point $(-p;q)$ and a point on the graph**	**Given three points (NOT x-intercepts or a turning point)**
$y=a(x-x_1)(x-x_2)$	$y=a(x+p)^2+q$	$y=ax^2+bx+c$
• Substitute x-intercepts, x_1 and x_2 into correct formula. • Replace *a* with the coefficient of x^2 if given, but if not given substitute the given point $(x;y)$ to find *a* • Multiply out the brackets to get the equation in standard form $y=ax^2+bx+c$	• Substitute the coordinates of turning point $(-p;q)$ into correct formula. • Replace *a* with the coefficient of x^2 if given, but if not given substitute the given point $(x;y)$ to find *a* • Multiply out the brackets to get the equation in standard form $y=ax^2+bx+c$	Determine the equation of the parabola if you are given three points: $(1;-2)$, $(2;-3)$ and $(0;3)$. • Identify the point representing the *y*-intercepts. • Subst. the *y*-intercept into the equation $y=ax^2+bx+c$ to get the *c*-value. • Now subst. the other two points into the equation and solve simultaneously to determine the values of *a* and *b*.

NOTATION			
$f(x)>0$	above the *x*-axis (line $y=0$)	 $x<-2$ or $x>2$	 $-2<x<2$
$f(x)<0$	below the *x*-axis (line $y=0$)	 $-2<x<2$	 $x<-2$ or $x>2$

$f(x).g(x) \geq 0$	Both graph above *x*-axis or both graph below *x*-axis	 $x \leq -2$ or $-1 \leq x \leq 2$	 $x \leq 2$ or $x \geq 1$
$f(x).g(x) \leq 0$	One graph above *x*-axis (line $y = 0$) and other one below *x*-axis (line $y = 0$)	 $-2 \leq x \leq -1$ or $x \geq 2$	 $-2 \leq x \leq 1$
$f(x) \geq g(x)$	*f* must be above *g*	 $x \leq -2.30$ or $x \geq 1.30$	 $-2.8 \leq x \leq 1.8$
$f(x) = g(x)$	Where the two graph intersect	$x = -2.8$ and $x = 1.8$	

Worked-out example 4

The graph of $f(x) = ax^2 + q$ is sketched below.
Points A(2 ; 0) and B(-3 ; 2,5) lie on the graph of *f*.
Points A and C are *x*-intercepts of *f*.



4.1	Write down the coordinates of C.	(1)
4.2	Determine the equation of *f*.	(4)
4.3	Write the down the range of *f*.	(1)
4.4	Write down the range of h, where $h(x) = -f(x) - 2$	(2)

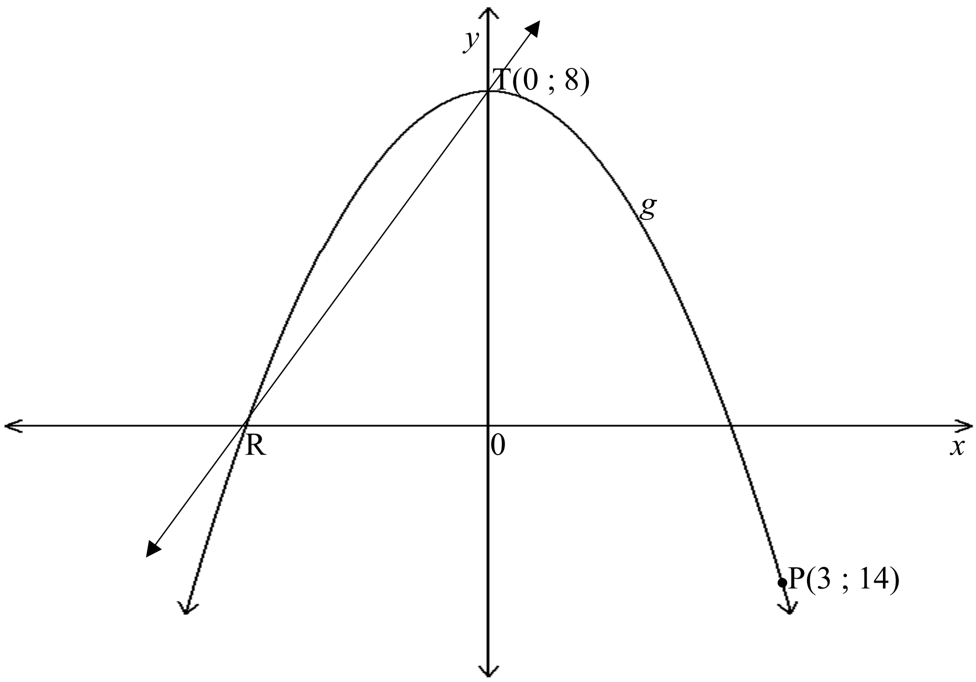
Possible solution

4.1	C(-2 ; 0)	✓ C(-2 ; 0) (1)
4.2	$y = a(x - x_1)(x - x_2)$ $y = a\left(x - (-2)\right)\left(x - 2\right)$ $y = a(x+2)(x-2)$ Subst. B(-3 ; 2,5) $2,5 = a(-3+2)(-3-2)$ $2,5 = a(-1)(-5)$ $2,5 = 5a$ $\therefore \frac{1}{2} = a$ $y = \frac{1}{2}(x+2)(x-2)$ $y = \frac{1}{2}(x^2 - 4) \;\; \therefore y = \frac{1}{2}x^2 - 2$	✓ subst. of *x* = -2 & *x* =2 ✓ subst. of B(-3;2,5) ✓ value of *a* ✓ equation (4)
4.3	y-intercept is -2	✓ critical value

	$\therefore y \in [-2;\infty)$	✓ notation (2)
4.4	$h(x) = -(\frac{1}{2}x^2 - 2) - 2$ $= -\frac{1}{2}x^2 + 2 - 2$ $= -\frac{1}{2}x^2$ $\therefore y \in (-\infty;0]$	✓ equation ✓ $y \in (-\infty;0]$ (2)

Worked-out example 5

The diagram shows the graphs of $g(x) = ax^2 + q$ and $f(x) = mx + c$.
R is the *x*-intercept of *g* and T(0 ; 8) is the *y*-intercept of g. P(3 ; 14) is the point of *g*
Graph f passes through R and T.



5.1 Write down the range of *g*. (2)

5.2 Determine the values of *a* and *q*. (2)

5.3 If coordinates of R are (-2 ; 0), use the graph to determine value(s) of x for which:

5.3.1 $f(x) < g(x)$ (2)

[06]

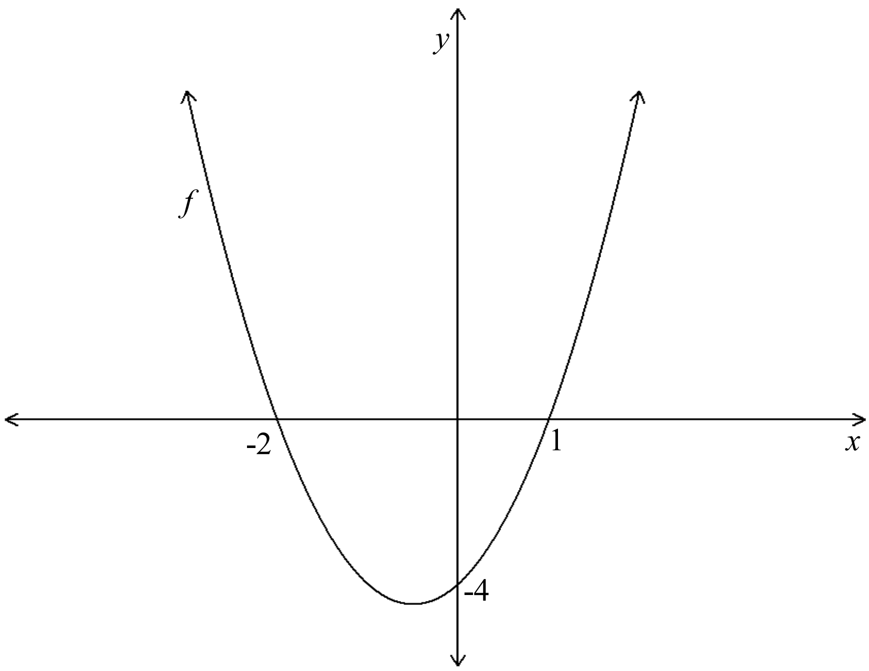
Possible solutions

5.1	$y \in (-\infty; 8]$	✓ interval ✓ notation (2)
5.2	$y = ax^2 + 8$ $14 = a(3)^2 + 8$ $6 = 9a$ $\therefore 2 = a$	✓ subst. of P ✓ value of *a* (3)
5.3	$-2 < x < 0$ $(-2; 0)$	✓ interval ✓ notation (2)

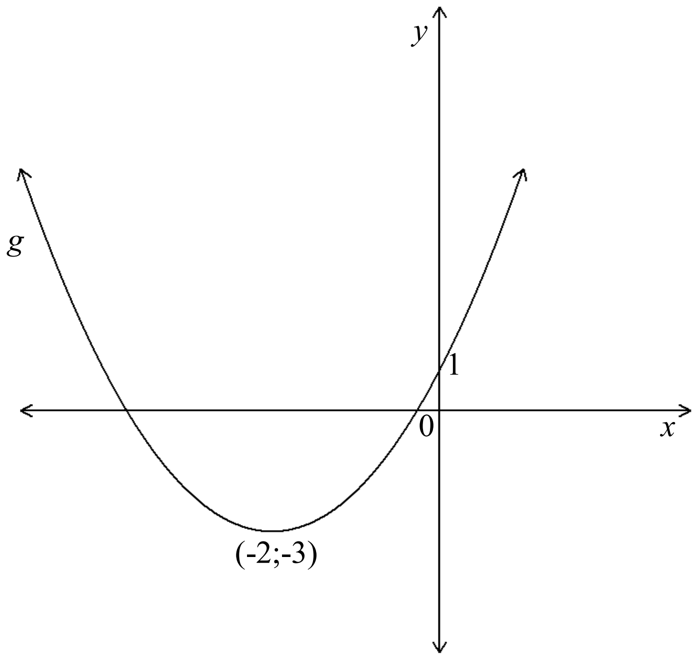
Worked-out example 6

Determine the equation of graph below in the form $y = ax^2 + b + c$

6.1



6.2



Possible solution

6.1	$y=a(x-x_1)(x-x_2)$ $y=a(x-(-2))(x-1)$ $y=a(x+2)(x-1)$ Subst. (0;-4) $-4=a(0+2)(0-1)$ $-4=a(2)(-1)$ $-4=-3a$ $\frac{4}{3}=a$ $y=\frac{4}{3}(x+2)(x-1)$ $y=\frac{4}{3}(x^2+x-2)$ $y=\frac{4}{3}x^2+\frac{4}{3}x-2$	✓ subst. of x = -2 & x =1 ✓ subst. of B(-3;2,5) ✓ value of a ✓ equation (4)
6.2	$y=a(x+2)^2-3$ Subst. (0;1) $1=a(0+2)^2-3$ $4=a(2)^2$ $4=4a$ $\therefore a=1$ $y=(x+2)^2-3$ $y=x^2+4x+4-3$ $y=x^2+4x+1$	✓ subst. of TP ✓ subst. of (0;1) ✓ value of a ✓ equation (4)

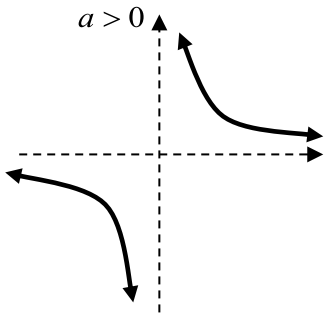
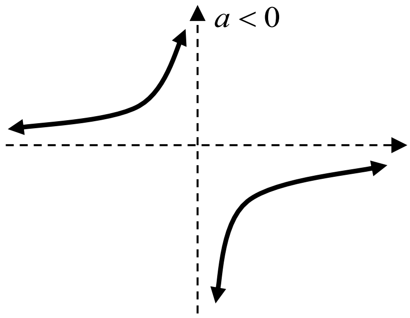
Hyperbolic Function

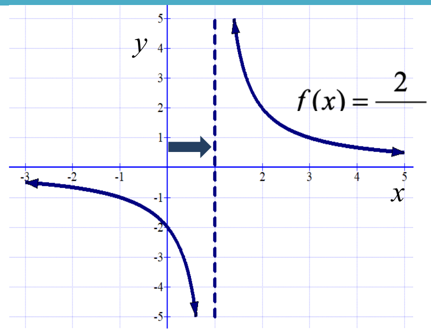
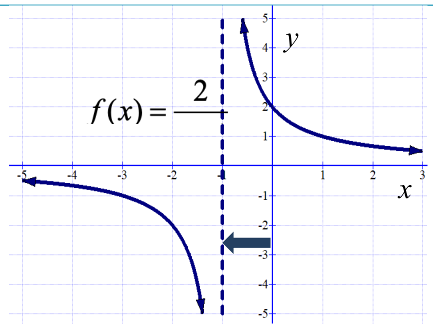
1. The general equation is given by: $y = \frac{a}{x} + q$

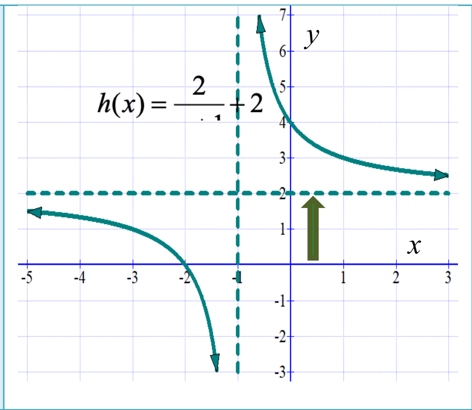
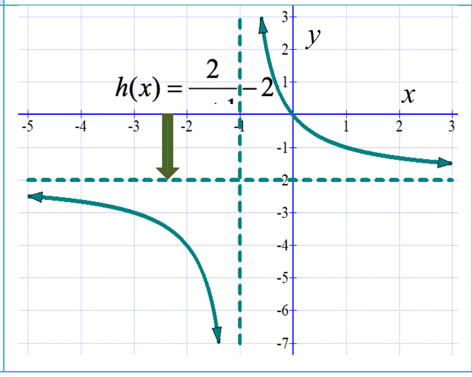
$: \; y = \frac{a}{x+p} + q$

q is the vertical shift

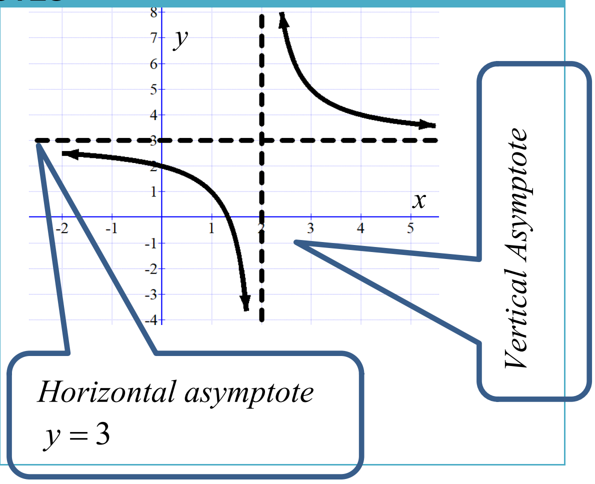
p is the horizontal shift

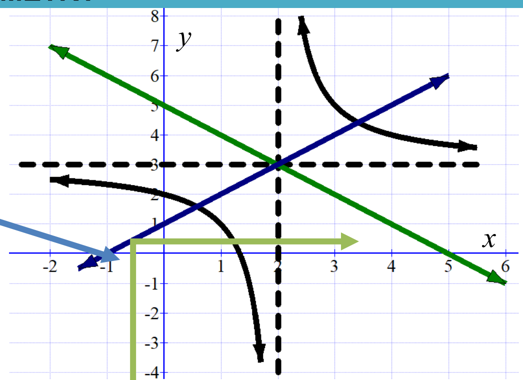
SHAPE	
Happy face parabola (**Concave up**)	Sad face parabola (**Concave down**)
The hyperbola will lie in the 1st and 3rd quadrants.	The hyperbola will lie in the 2nd and 4th quadrants
	
If $a > 0$, then the graph decreases for all *x* values. Where $x \in \mathbb{R}\,; x \neq -p$	If $a < 0$, then the graph increases for all *x* values. Where $x \in \mathbb{R}\,; x \neq -p$

EFFECTS OF *p* AND *q*		
$p < 0$ (negative) ***p*** affects the **horizontal shift (***moves graph to the right*) of the graph and gives the position of the vertical asymptote $(x = -p)$.		The graph moves *p* units to the left
$p > 0$ (positive) ***p*** affects the **horizontal shift (***moves graph to the left*) of the graph and gives the position of the vertical asymptote $(x = -p)$.		The graph moves *p* units to the right

$q > 0$ (positive) ***q*** affects the **vertical shift** of the graph. Also gives the horizontal asymptote $y = q$		The graph moves q units **upwards**.
$q < 0$ (negative) ***q*** affects the **vertical shift** of the graph. Also gives the horizontal asymptote $y = q$		The graph moves q units **downwards.**

INTERCEPTS	
Intercepts	**example**
x – intercept; $y = 0$	$y = \frac{3}{x-2} + 3$ let $y = 0$ $0 = \frac{3}{x-2} + 3$ $-3 = \frac{3}{x-2}$ $-3(x-2) = 3$ $-3x + 6 = 3$ $\therefore x = 1$
y – intercept; $x = 0$	$y = \frac{3}{x-2} + 3$ Let $x = 0$ $y = \frac{3}{0-2} + 3$ $y = \frac{3}{2}$

ASYMPTOTES		
The graph has **Two asymptotes:** • Vertical asymptote - $x=-p$ • Horizontal asymptote - $y=q$	Example: $y=\dfrac{2}{x-2}+3$ Vertical asymptote: $x-2=0$ $\therefore x=2$ Horizontal asymptote: $y=3$	

AXIS OF SYMMETRY		
• For increasing or $m>0$, use $y=x+c$ or $y=(x+p)+q$ • For decreasing or $m<0$, use $y=-x+c$ or $y=-(x+p)+q$ ***NB: Both axis of symmetry pass through point of intersection of two asymptotes***	**Point of intersection (2;3)** $3=2+c$ $\therefore c=1$ $\therefore y=x+1$ or $y=(x-2)+3$ $y=x-2+3$ $\therefore y=x+1$ $3=-2+c$ $\therefore c=5$ $y=-x+5$ or $y=-(x-2)+3$ $y=-x+2+3$ $\therefore y=-x+5$	

DOMAIN AND RANGE	
Example: $y=\frac{2}{x-2}+3$	
Domain: $x\in\mathbb{R}\,;x\neq -p$	$x\in\mathbb{R}\,;x\neq 2$ or $x\in(-\infty;\infty);x\neq 2$
Range: $y\in\mathbb{R}\,;y\neq q$	$y\in\mathbb{R}\,;y\neq 3$ or $y\in(-\infty;\infty);y\neq 3$

Sketching of hyperbola

i) Determine the shape.

ii) Write down the equations of asymptotes.

iii) Determine *x*- and *y*- intercepts.

iv) Sketch the graph starting with asymptotes.

Worked-out example 1

Given $f(x)=\frac{2}{x}-3$

1.1 Write down the equations of the asymptotes of *f*. (2)

1.2 Write down the domain and range of *f*. (2)

1.3 Calculate the *x*-intercepts of the graph of *f*. (2)

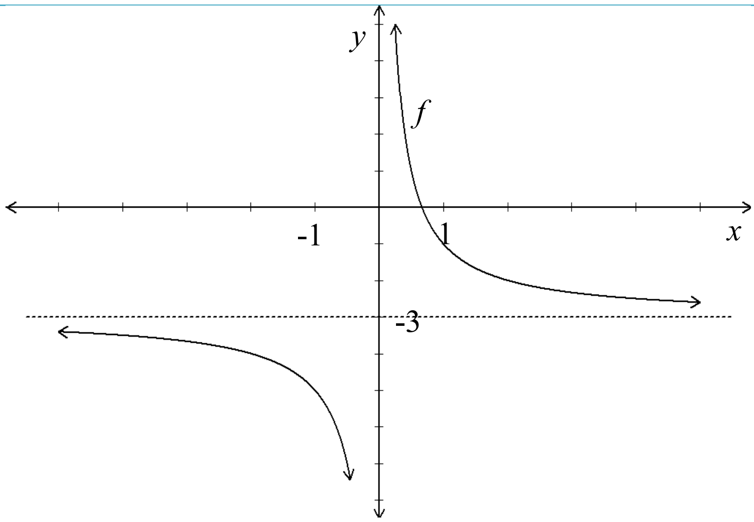
1.4 Draw a neat sketch of *f*, clearly showing all intercepts with the axes and asymptotes. (3)

1.5 Determine the equation of the axis of symmetry that has positive gradient (2)

[10]

Possible solutions

1.1	$x=0$ $y=-3$	✓ $x=0$ ✓ $y=-3$ (2)
1.2	$x\in\mathbb{R}\,;x\neq 0$ $y\in\mathbb{R}\,;y\neq -3$	✓ $x\in\mathbb{R}\,;x\neq 0$ ✓ $y\in\mathbb{R}\,;y\neq -3$ (2)
1.3	$0=\frac{2}{x}-3$ $3=\frac{2}{x}$ $3x=2$ $\therefore x=\frac{2}{3}$	✓ equating *y* to 0 ✓ answer (2)

1.4		✓ *x* & *y*- intecepts ✓ asymptote ✓ shape (3)
1.5	$y = x + q$ $-3 = 0 + q$ $-3 = q$ $\therefore y = x - 3$	✓ subst. (0;-3) ✓ equation (2)

Worked-out example 2

Given $h(x) = \dfrac{3}{x-1} + 2$

2.1 Determine the equations of the asymptotes. (2)

2.2 Calculate *x*- and *y*-intercepts of *h.* (3)

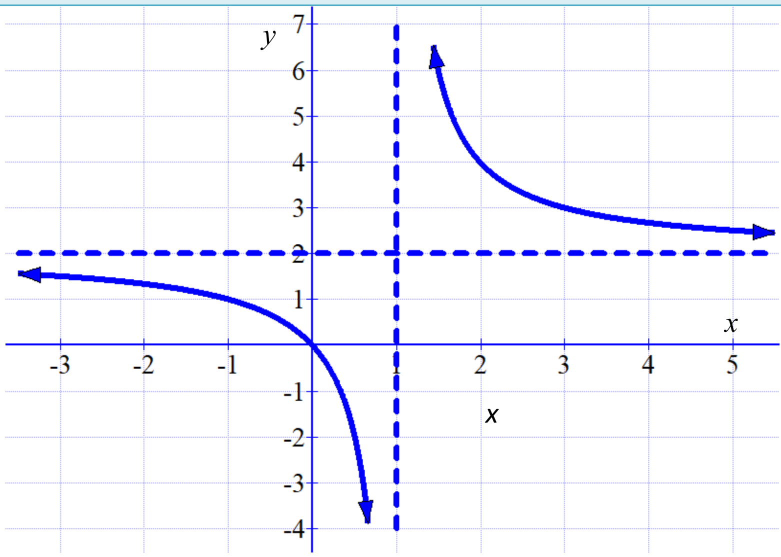
2.3 Sketch the graph of h, clearly show all intercepts with asymptotes. (3)

2.4 Determine range of *h.* (2)

2.5 Determine the value(s) of *x* for which $h(x) \leq 0$ (2)

2.6 Show that the equation of axis of symmetry with positive gradient is given by $y = -x + 3$.
(2)
[14]

Possible solution

2.1	$x = 1$ $y = 2$	✓ $x = 1$ ✓ $y = 1$ (2)
2.2	$y = \dfrac{2}{0-1} + 2$ $= 0$ *x*-intercept; let $y = 0$ $0 = \dfrac{2}{x-1} + 2$ $-2 = \dfrac{2}{x-1}$ $-2(x-1) = 2$ $-2x + 1 = 2$ $x = -\dfrac{1}{2}$	✓ $y = 0$ ✓ $y = 0$ ✓ $x = -\dfrac{1}{2}$ (3)
2.3		✓ point (0;0) ✓ asymptotes ✓ shape (3)
x2.4	$y \in \mathbb{R}\,; y \neq 2$	✓✓ answer (2)
2.5	$0 \leq x < 1$ or $x \in [0;1)$	✓ critical values ✓ notation
2.6	$y = -x + c$ $2 = -1 + c$ $3 = c$ $\therefore y = -x + 3$	✓ negative gradient ✓ subst. (1;2) (2)

DETERMINING EQUATIONS OF A HYPERBOLIC FUNCTIONS

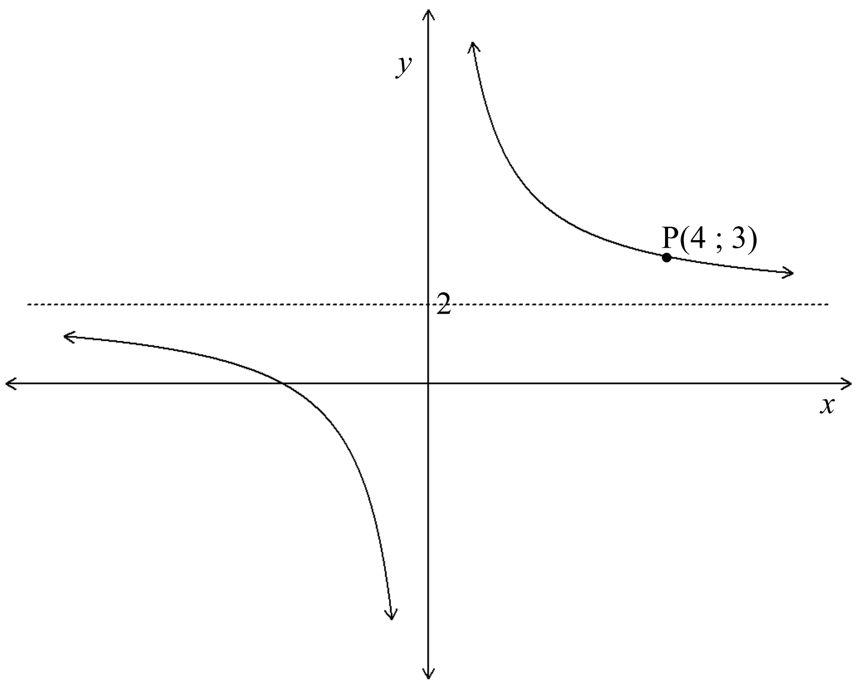
NB: Given the asymptote and a point on the graph.

- Determine the equations of the asymptotes,
- Determine the values of *p* and *q,* then substitute into the equation.
- Substitute coordinates of a point on the graph into the equation to determine the value *a*.
- Write down the final equation in the form $y=\dfrac{a}{x+p}+q$

Worked-out example 1

The diagram below represents the graph of $h(x)=\dfrac{a}{x}+q$.

P(4 ; 3) is a point on f and equation of asymptote is y = 2



1.1	Determine the equation of *h.*	(3)
1.2	Calculate the coordinates of *x*-intercept of *h.*	(2)
1.3	Determine from the graph the value(s) of *x* for which:	
	1.3.1 $h(x)\leq 0$	(2)
	1.3.2 *h* is increasing	(1)
1.4	Calculate the range of *p*, if $p(x)=-h(x)+1$.	(2)

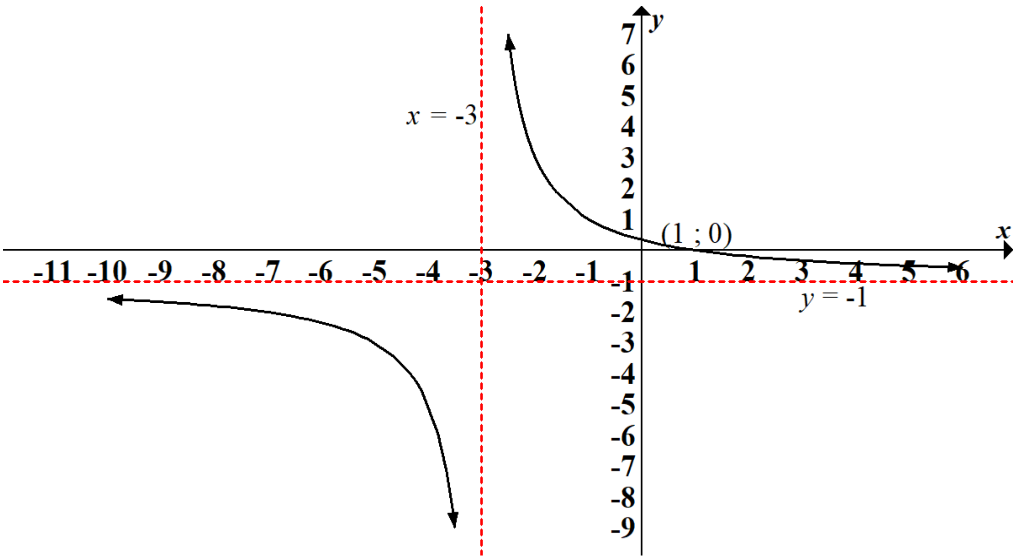
[11]

Possible solutions

2.1	$y=\frac{a}{x}+2$ $3=\frac{a}{4}+2$ $1=\frac{a}{4}$ $a=4$	✓ $q = 2$ ✓ subs. (4 ; 3) ✓ value of a (3)
2.2	$y=\frac{4}{x}+2$ $0=\frac{4}{x}+2$ $-2=\frac{4}{x}$ $-2x=4$ $\therefore x=-2$	✓ $y = 0$ ✓ answer (2)
2.3.1	$-2 \leq x < 0$ or $x \in [-2;0)$	✓ interval ✓ notation (2)
2.3.2	$x \in \mathbb{R}\,; x \neq 0$	✓ answer (1)
2.4	$p(x)=-\left(\frac{4}{x}+2\right)+1$ $p(x)=-\frac{4}{x}-2+1$ $p(x)=-\frac{4}{x}-1$ $\therefore y \in \mathbb{R}\,; y \neq -1$	✓ equation ✓ answer (2)

Worked-out example 2

Determine the equation of the graph drawn below. (4)



[4]

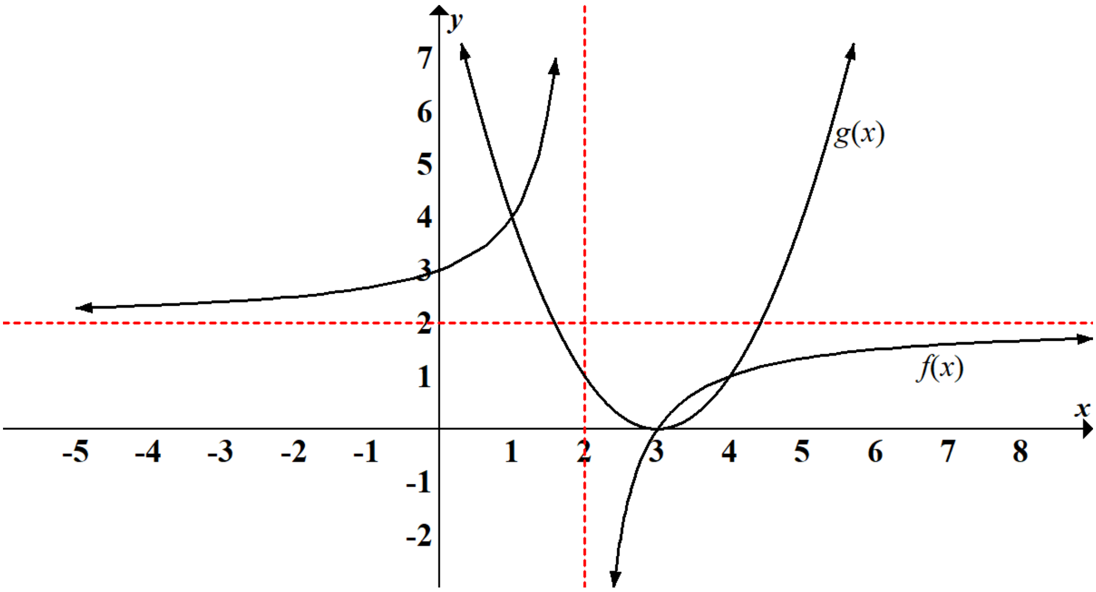
Possible solutions

2.1	The vertical asymptote is $x=-3$ The horizontal asymptote is the line $y=-3$ $\therefore\ y=\dfrac{a}{x-(-3)}-1$ $y=\dfrac{a}{x+3}-1$ Subst, the point $(1;0)$ to find the value of *a*. $0=\dfrac{a}{1+3}-1$ $1=\dfrac{a}{4}$ $a=4$ Therefore, the equation of the graph is $y=\dfrac{4}{x+3}-1$	✓ **Subst. of asymptotes** ✓ **Subst. point** $(1;0)$ ✓ $a=4$ ✓ **answer** $\left(y=\dfrac{4}{x+3}-1\right)$ (4)

Worked-out example 3

The graph below shows the graphs of the function $f(x)=\dfrac{a}{x+p}+q$ and $g(x)=(x-3)^2$.

The *y*-intercept of $f(x)$ is $(0;3)$



1.1.	Write down the equation of the asymptotes of $f(x)$.	**(2)**
1.2.	Determine the values of *a*, *p* and *q*.	**(4)**
1.3.	What are the coordinates of the turning point of $g(x)$.	**(1)**
1.4.	What is the range of $g(x)$**?**	**(1)**
1.5.	What is the domain of $f(x)$**?**	**(2)**
1.6.	Give one value of *x* for which $f(x)=g(x)$.	**(2)**
1.7.	If $f(x)$ **is** shifted downwards by 4 units and to the left by 1 unit, give its new equation.	**(1)**
1.8.	For which values of *x* is $g(x)$ **increasing?**	**(1)**

[14]

Possible solutions

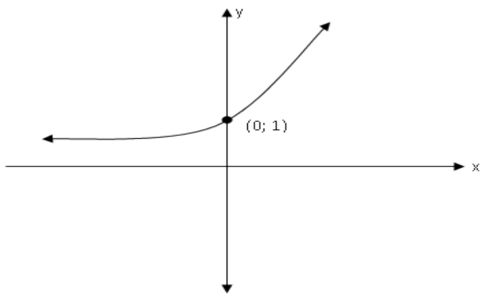
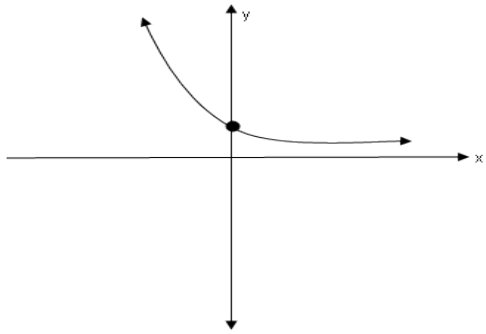
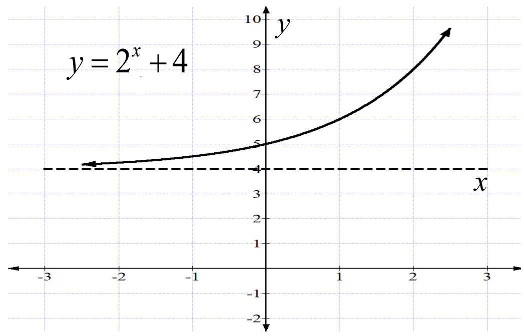
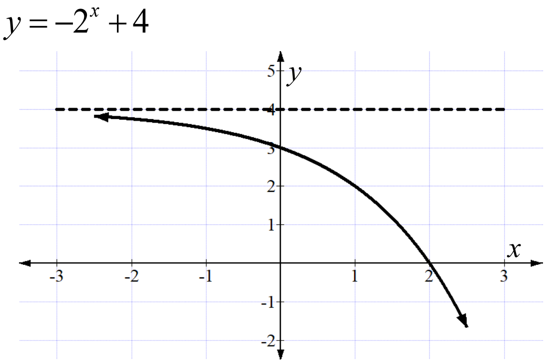
3.1	$x=2$ and $y=2$	✓ $x=2$ ✓ $y=2$ **(2)**
3.2	$p=2$ **and** $q=2$ $y=\dfrac{a}{x-2}+2$ To determine the value of *a*, subst. the point $(0;3)$ into the equation: $0=\dfrac{a}{3-2}+2$ $-2=\dfrac{a}{1}$ $a=-2$	✓ $p=2$ ✓ $q=2$ ✓ Subst $(0;3)$ ✓ $a=-2$ (4)
3.3	The turning point of $g(x)$ is $(3;0)$	✓ $(0;3)$ (1)
3.4	The range of $g(x)$ is $y\geq 0$	✓ $y\geq 0$ (1)
3.5	The domain of $f(x)$ is $x\in\mathbb{R}'$, $x\neq 2$	✓✓ $x\in\mathbb{R}'$, $x\neq 2$ (2)
3.6	The two graphs intersect when $x=3$ (and when $x=1$)	✓ $x=3$ ✓ $x=1$ (2)
3.7	$y=\dfrac{2}{x-1}-2$	✓ $y=\dfrac{2}{x-1}-2$ (1)
3.8	$g(x)$ is increasing for $x>3$.	✓ $x>3$ (1)

Exponential function

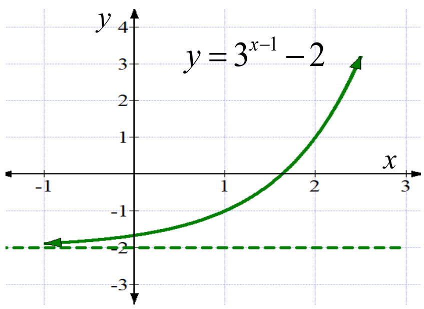
1. Standard form: $f(x) = a.b^x + q$

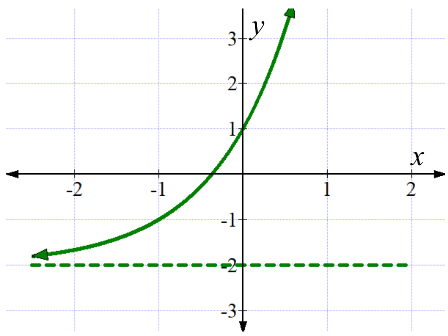
: $y = a.b^{x+p} + q$, where $b > 0$, $b \neq 1$

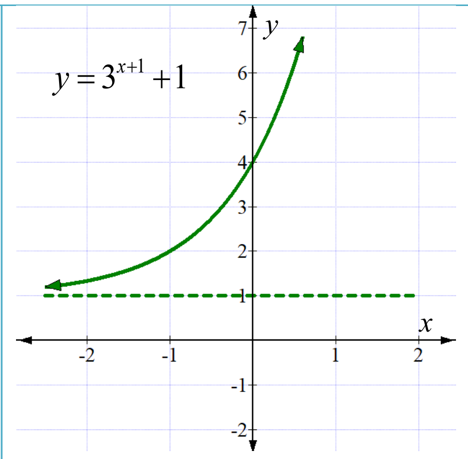
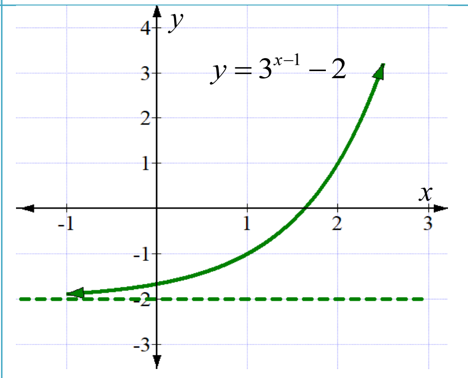
SHAPE

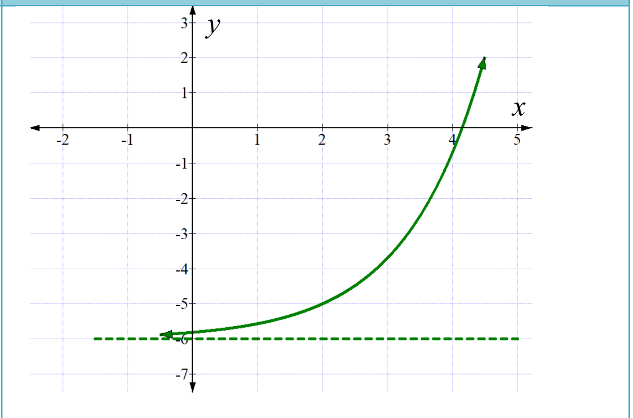
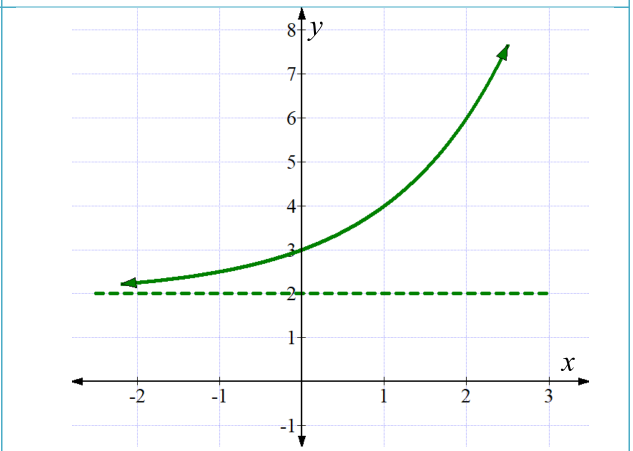
<table>
<tr><td>
</td><td>$0 < b < 1$

</td></tr>
<tr><td colspan="2">a determines the shape of the graph.</td></tr>
<tr><td>If $a > 0$ then the graph lies above the asymptote.

</td><td>If $a < 0$ then the graph lies below the asymptote.

</td></tr>
</table>

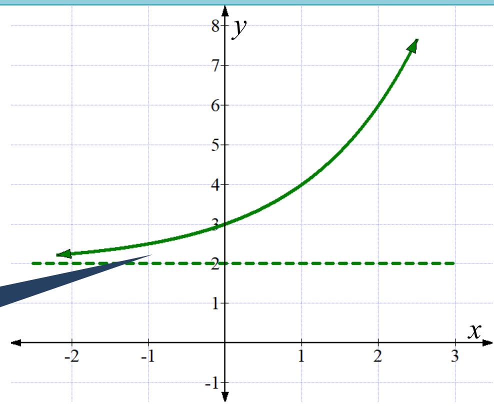
EFFECTS OF *p* AND *q*

<table>
<tr><td>p affects the horizontal shift of the graph.</td><td rowspan="2">


</td><td>If $p < 0$ (negative), then the graph moves p units to the right</td></tr>
<tr><td>p affects the horizontal shift of the graph.</td><td>If $p > 0$ (positive), then the graph moves p units to the left</td></tr>
</table>

q affects the **vertical shift** of the graph. Also gives the horizontal asymptote $y=q$		If $q>0$ (positive), then the graph moves *q* units **upwards**.
q affects the **vertical shift** of the graph.		If $q<0$ (negative), then the graph moves *q* units **downwards.**

INTERCEPTS WITH AXIS		
Intercepts	**example**	**Graph**
x-intercept; let $y=0$	Given: $f(x)=2.3^{x-2}-6$ let $y=0$ $0=2.3^{x-2}-6$ $6=2.3^{x-2}$ $3=3^{x-2}$ $\therefore x-2=1$ $x=3$	
	Given: $h(x)=2^{x}+2$ let $y=0$ $0=2^{x}+2$ $-2=2^{x}$	

ASYMPTOTE		
The graph has **ONLY ONE asymptote:** • Horizontal asymptote - $y=q$	Example $h(x)=2^x+2$ Horizontal Asymptote $y=2$ Horizontal asymptote	
NB: Exponential graph **does not have a vertical asymptote**.		

DOMAIN AND RANGE	
The **domain** of an exponential graph is $x\in\mathbb{R}$.	Example $f(x)=2.3^{x-2}-6$ Domain: $x\in\mathbb{R}$
The **range** of an exponential graph is determined by the value of ***q*.** $y\in(q;\infty)$, if $a>0$ $y\in(\infty;q)$, if $a<0$	$f(x)=2.3^{x-2}-6$ Range: $a>0$ $y\in(-6;\infty)$ Range: $a<0$ $h(x)=-3.\left(\frac{1}{2}\right)^x+2$ $y\in(\infty;2)$

Sketching of exponential graphs

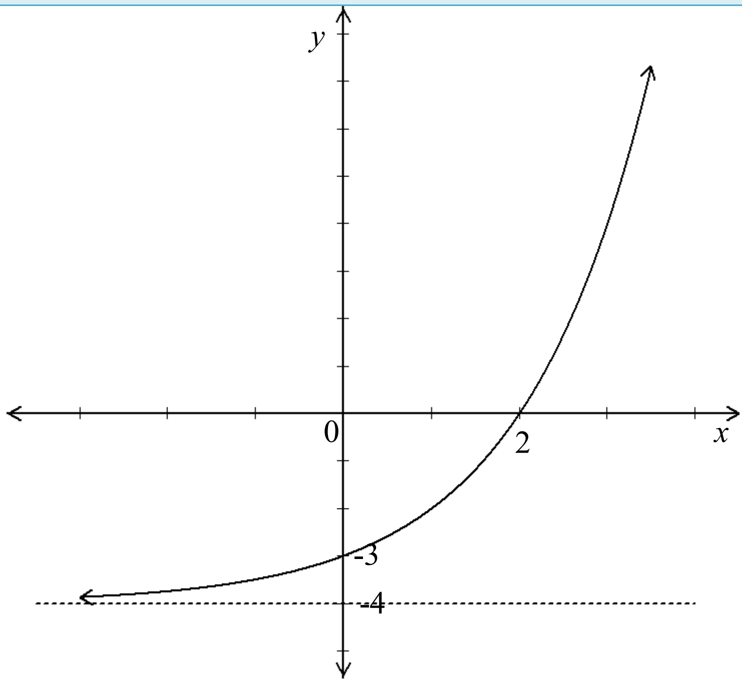
- Draw the asymptote using dotted line **(the asymptote of** $y=a.b^{x+p}+q$ **is** $y=q$ **).**
- Calculate the *x*-intercept and *y*-intercept and plot them on the axes.
- If necessary, calculate more points to enable you to draw the correct curve.
- Draw the curve and add the equation of the graph to both curves.

Worked-out example 1

Given: $h(x) = 2^x - 4$

1.1	Write down the equation of asymptote.	(1)
1.2	Determine the coordinates of the *y*-intercept of *h*.	(2)
1.3	Calculate the *x*-intercept of *h*.	(2)
1.4	Draw a sketch graph of *h*, clearly showing all asymptotes, intercepts with the axes and at least one other point on *h*.	(3)
1.5	Write down range of *h*.	(1)
1.6	Describe the transformation from *h* to *g* if $g(x) = 2^{-x} + 1$	(2)
		[11]

Possible solutions

1.1	$y = -4$	✓ $y = -4$ (3)
1.2	$h(x) = 2^0 - 4$ $y = -3$ $\therefore (0;-3)$	✓ $x = 0$ ✓ $y = 3$ (2)
1.3	$0 = 2^x - 4$ $4 = 2^x$ $2^2 = 2^x$ $\therefore 2 = x$ $(2;0)$	✓ $y = 0$ ✓ $x = 2$ (2)
1.4		✓ *x*- & *y*-intercepts ✓ asymptote ✓ shape (1)
1.5	$y \in (-4;\infty)$	✓ answer

1.6	$g(x)=h(-x)+1$ $g(x)=2^{-x}-4+5$ $g(x)=2^{-x}+1$ $\therefore$ Reflection about **the *y*-axis** and **shifted five** units up.	✓ the *y*-axis ✓ shifted five (2)

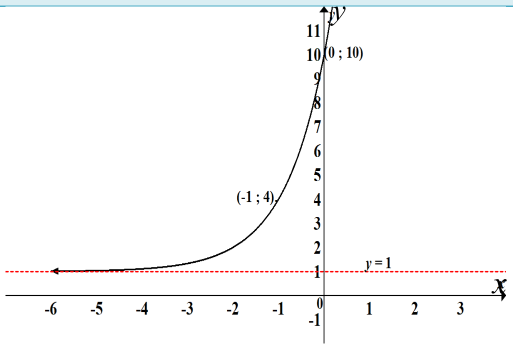
Worked-out example 2

2.1 Sketch the graph of $f(x)=3^{x+2}+1$ (4)

2.2 Label all intercepts with the axes and asymptotes. (3)

2.3 Write down the domain and range of the graph. (2)

Possible solutions

2.1	The horizontal asymptote is $y=1$. For the *x*-intercept, let $y=0$: $0=3^{x+2}+1$ $-1=3^{x+2}$ There are no *x*-intercept. For the *y*-intercept, let $x=0$: $y=3^{0+2}+1$ $y=9+1$ $=10$ $y-\text{intercept } is \ (0;10)$ To plot another point on the graph, when $x=-1$: $y=3^{-1+2}+1$ $y=4$ $\therefore \text{Point } (-1;4) \text{ is on the graph}$	✓ Subst. asymptotes ✓ Simplification $(-1=3^{x+2})$ ✓ *y*-intercept ✓ $\text{point } (-1;4)$ (4)
2.2		✓ shape ✓ *y*-intercept ✓ asymptote (3)
2.3	The domain: $x\in\mathbb{R}$ The range: $y>1$	✓ The domain: $x\in\mathbb{R}$ ✓ The range: $y>1$ (2)

Worked-out example 3

Write down the equation of the function $f(x)=2^x$, in the form $y=\ldots$, if it is:

3.1 Shifted to the right by 3 units. (1)

3.2 Shifted to the right by 2 units and upwards by 1 unit. (1)

3.3 Shifted downwards by 3 units and to the right by 1 unit. (1)

3.4 Stretched vertically by 3 units and shifted to the left by 2 units. (1)

Possible solutions:

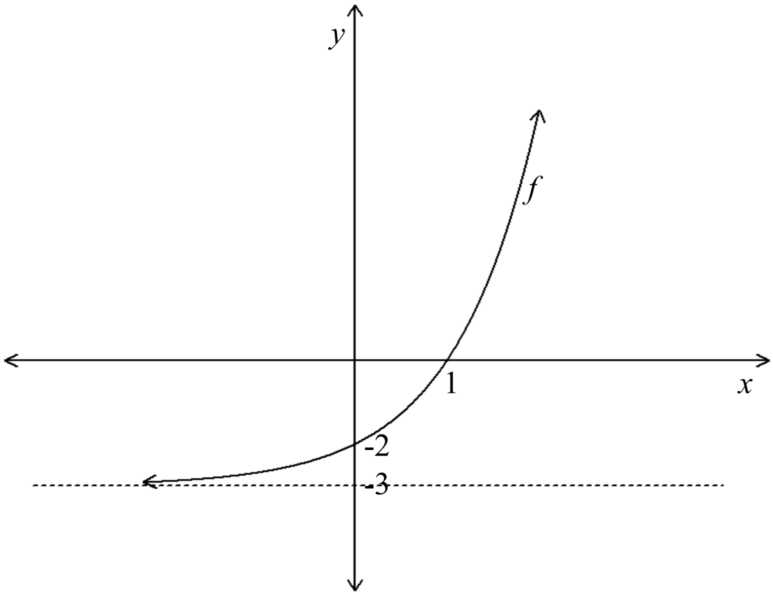
3.1	$y=2^{x-3}$	✓ Answer (1)
3.2	$y=2^{x+2}+1$	✓ Answer (1)
3.3	$y=2^{x-1}-3$	✓ Answer (1)
3.4	$y=3.2^{x+2}$	✓ Answer (1)

Determining the equation

- Determine the equation of the asymptotes,
- Determine the value of *q,* then substitute into the equation.
- Substitute coordinates of a point on the graph into the equation to determine the value *a*.
- Write down the final equation in the form $y=a.b^{x+p}+q$

Worked-out example 4

Sketched below is the graph of $f(x) = b^x + m$ passing through $(0\,;-2)$ and $(1\,;0)$.
The equation for asymptote of f is given by the equation $y = -3$.



4.1	Determine the value of *m* and *b*.	(3)
4.2	Write down the range of *f*.	(1)
4.3	For which values of *x* will $f(x) = -\frac{26}{9}$	(3)
		[07]

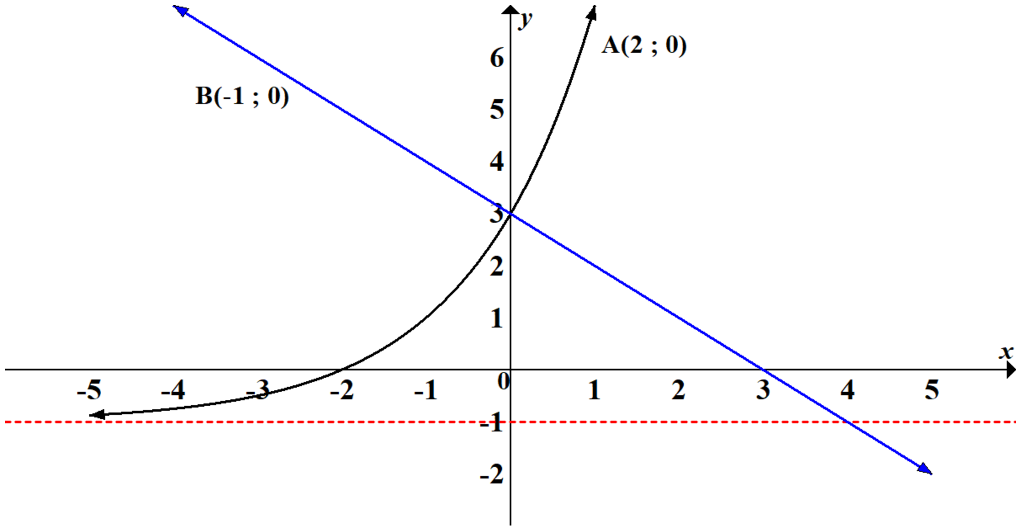
Possible solution

4.1	$m = -3$ $y = b^x - 3$ $0 = b^1 - 3$ $\therefore 3 = b$	✓ $m = -3$ ✓ subst. (1 ; 0) ✓ value of *b* (3)
4.2	$y \in (-3;\infty)$	✓ answer (1)
4.3	$3^x - 3 = -\frac{26}{9}$ $3^x = 3 - \frac{26}{9}$ $3^x = \frac{1}{9}$ $3^x = \frac{1}{3^2}$ $3^x = 3^{-2}$	✓ subst. $-\frac{26}{9}$ ✓ simplification ✓ answer

	$\therefore x = -2$	(3)

Worked-out example 5

The diagram below shows the graphs of $f(x) = 2^{x+p} + q$ and $g(x) = -x + c$



4.1 Determine the values of *p*, *q* and *c*. (5)

4.2 Write down the range of $f(x)$. (1)

4.3 For which value (s) of x is:

4.3.1 $f(x) = g(x)$? (1)

4.3.2 $f(x) \geq g(x)$? (1)

4.3.3 $f(x) \leq g(x)$? (1)

4.4 If $f(x)$ is shifted to the right and 3 values up, write down its new equation. (3)

[12]

Possible solutions

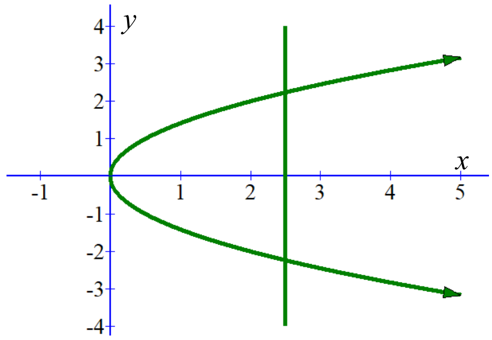
5.1	$c=3$ (y-intercept, from the sketch). $q=-1$ (Asymptote of the exponential graph) Subst. the point $(-2;0)$ into the equation to determine the value of *p*. $0=2^{-2-p}-1$ $1=2^{-2-p}$ $2^0=2^{-2-p}$ (exponential law) $\therefore 0=-2-p$ $p=-2$	✓ $c=3$ ✓ $q=-1$ ✓ subst. $(-2;0)$ ✓ exponential laws ✓ $p=-2$ (5)
5.2	The range $f(x)$ is $y>-1$.	✓ $y>-1$ (1)
5.3.1	$f(x)=g(x)$ when $x=0$ **(Point of intersection)**	✓ $x=0$ (1)
5.3.2	$f(x)\geq g(x)$ when $x\geq 0$ **(** $f(x)$ **lies above** $g(x)$**)**	✓ $x\geq 0$ (1)
5.3.3	$f(x)\leq g(x)$ when $x\leq 0$ **(** $f(x)$ **lies below** $g(x)$**)**	✓ $x\leq 0$ (1)
5.4	$y=2^{x+2-1}-1+3$ (Subtract 1 from the p-value and add 3 to the q-value) $y=2^{x+1}+2$	✓ subtracting 1 ✓ adding 3 ✓ answer (3)

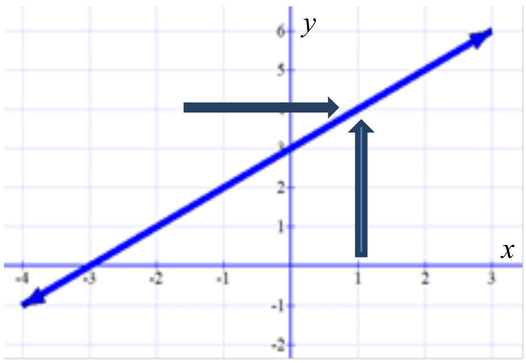
Inverse function

Learners must be able to determine, and sketches graph of inverse defined by:

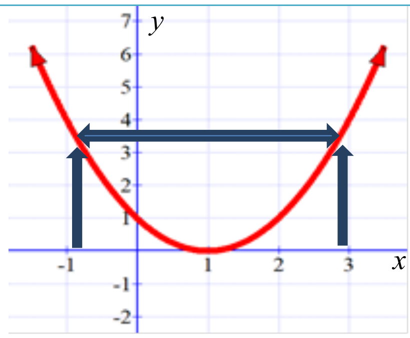
- $y = ax + q$
- $y = ax^2$
- $y = a^x$

THINGS TO REMEMBER
Definition of a function: A function *f*, is defined as a relationship between values, where each input value maps to one output value. (In other words: for an equation to be called a function there can be only one *y*-value for a particular *x*-value)

VERTICAL LINE TEST	
To test if a graph is a function, you can use the vertical line test: If a vertical line (a line parallel to the *y*-axis) touches the graph more than once at any point, the graph is not a function. ***NB: You do not have to draw the line physically; you can just hold a ruler parallel to the y-axis and move it for testing***.	

TWO TYPES OF FUNCTION	
One-to-one function: it is a function where there is a single *y*-value for a particular *x*-value	

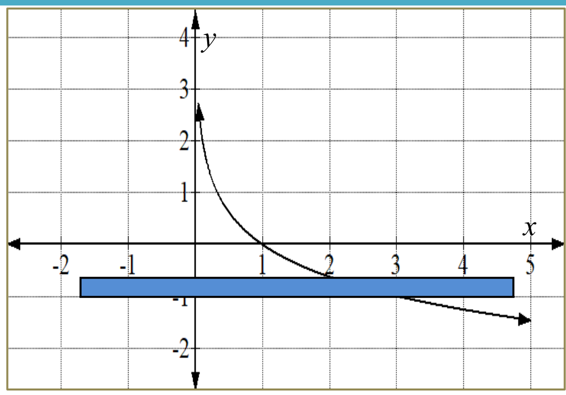
Many-to-one function:
A function cannot have more than one y value to each *x*-value. However, a function can have more than one *x*-value for a particular *y*-value.



HORIZONTAL LINE TEST

If the graph is passing the vertical line test, then it is a function. The horizontal line test can be used to determine what type of function the graph represents.

- If a horizontal line touches the graph once, it is one - to - one function.
- If a horizontal line touches the graph more than once at any point, it is a many-to-one function (many *x*-values to a single *y*-value).

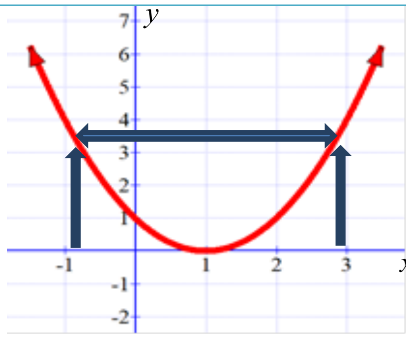
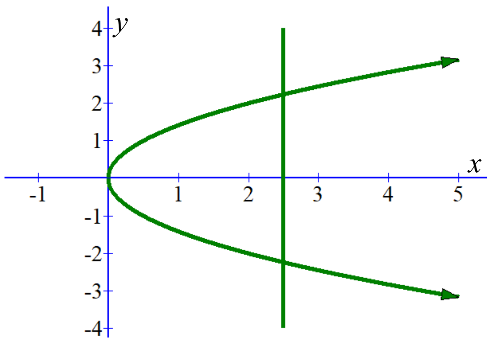
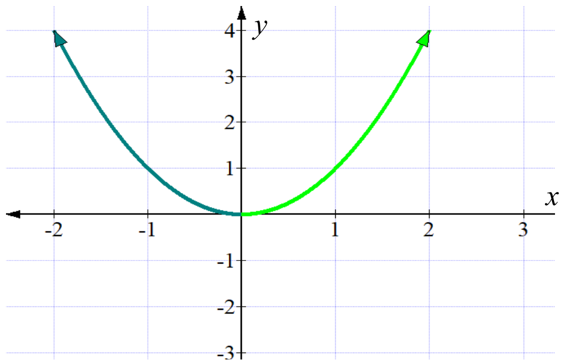
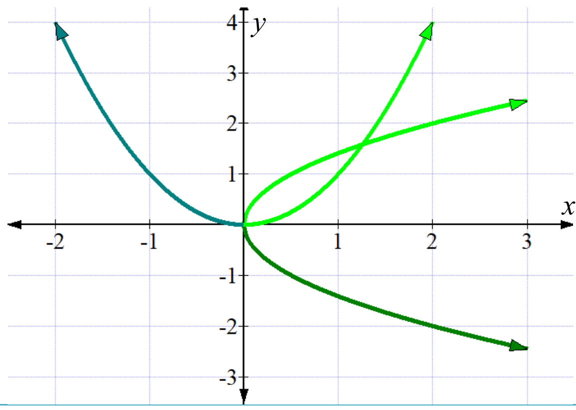


<u>Remember:</u> If a graph is not a function, it is known as a relation.

- To determine the inverse of a graph, swop every *x* and *y*-coordinate with each other. This is done for every point on the graph.
- To express this in the form of an equation, follow the same principal and swap the positions of *x* and *y*.
- In most cases the equation of a graph is given in the form of $f(x) = \ldots$
- The general way of expressing the inverse is in the form $f^{-1}(x)$.

The notation $f^{-1}(x) = \ldots$ can only be used to represent the inverse of a one-to-one relation.

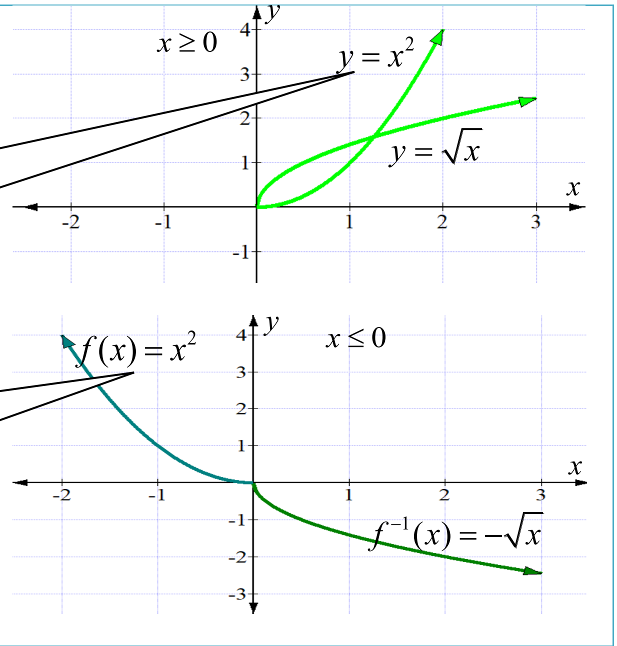
- It is important to notice that the domain and range if the inverse graph also swop.

RESTRICTING THE DOMAIN	
If a function is a many-to-one function (many *x*-values to a single *y*-value) the inverse will be a one-to-many relation (many *y*-values to a single *x*-value)	
This means that the inverse of any many-to-one function will never be a function.	
However, the domain of a many-to-one function can be restricted so that its inverse is a function	
Only the inverse of a one-to-one function is itself a function.	

A restriction in the domain of the original function corresponds to a restriction in the range of the inverse.

f:	f^{-1}
Domain: $x \geq 0$	Domain: $x \geq 0$
Range: $y \geq 0$	Range: $y \geq 0$

f:	f^{-1}
Domain: $x \leq 0$	Domain: $x \geq 0$
Range: f^{-1}	Range: $y \leq 0$



Worked-out example 1

Given: $f(x) = 2x + 6$

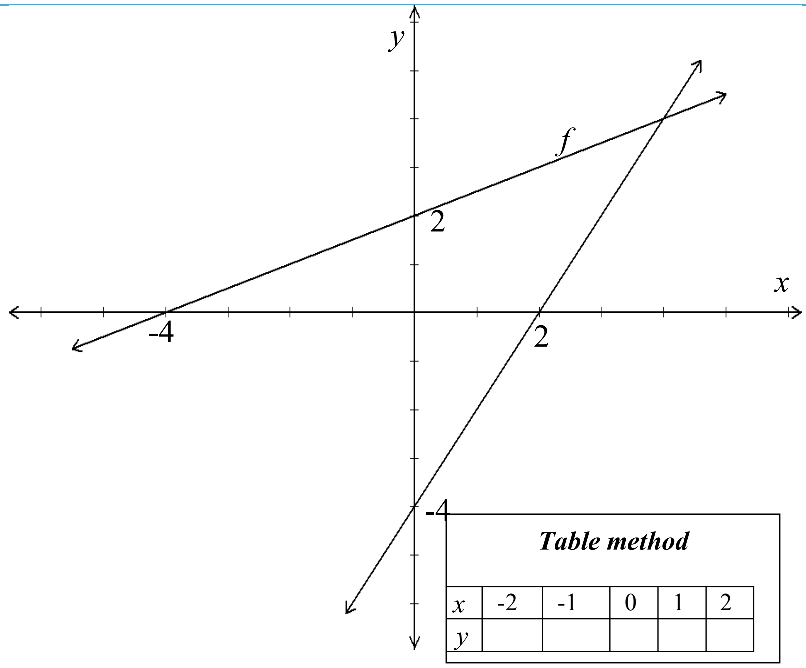
1.1 Determine the equation of the inverse of *f* in the form $y = ...$ (2)

1.2 On the same set of axis, sketch the graphs of *f* and f^{-1}. Indicate clearly the intercepts with the axis, as well as another point on the graph of each f and f^{-1}. (4)

1.3 Hence or otherwise, determine the *x* coordinate at the point of intersection of *f* and f^{-1}. (2)

Possible solutions

1.1	$y = 2x - 4$ $x = 2y - 4$ Swap x and y $2y = x + 4$ Make y subject of the formula $y = \frac{1}{2}x + 2$	✓ swapped *x* and *y* ✓ equation (2)

<table>
<tr><td>1.2</td><td></td><td>f
✓ intercepts
✓ shape

f^{-1}

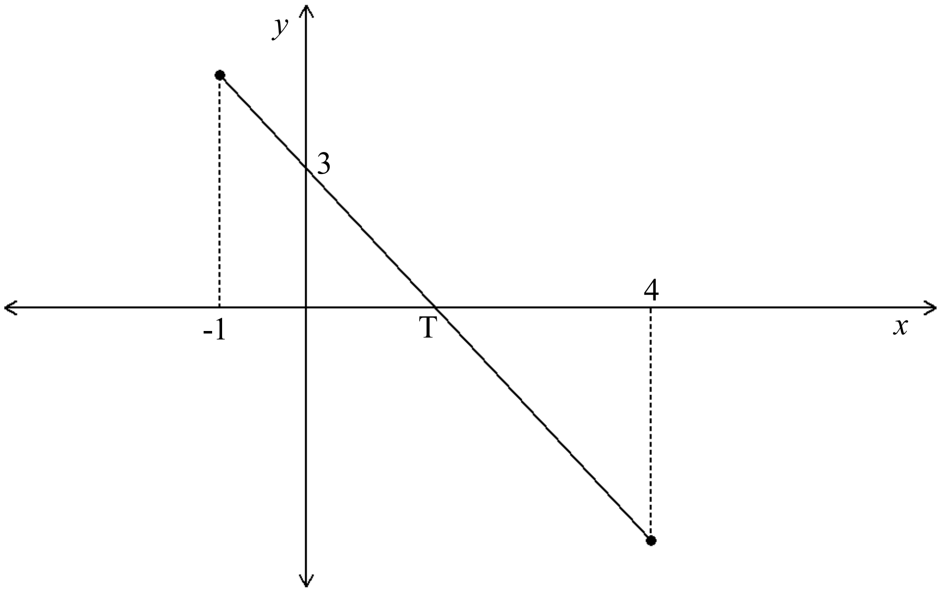
✓ intercepts
✓ shape

(4)</td></tr>
<tr><td>1.3</td><td>Read from the graph: x = 4</td><td>✓ ✓ answer

(2)</td></tr>
</table>

Worked-out example 2

Given: $g(x) = 2x + 3$ for $-1 \leq x \leq 4$.



2.1 Write down coordinates of T. (2)

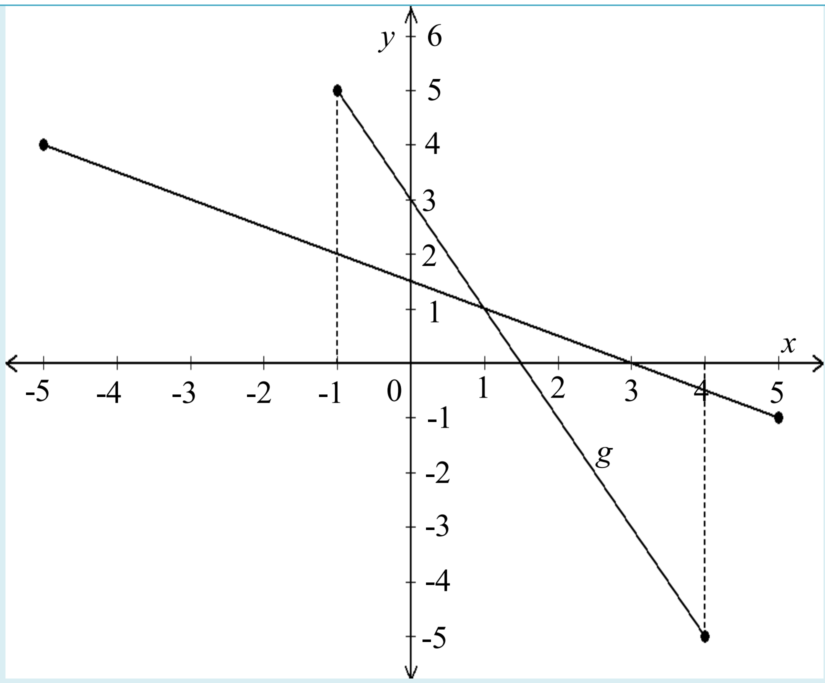
2.2 Determine the domain of g^{-1}. (2)

2.3 Sketch the graph of the g^{-1}. (3)

2.4 For which value(s) of *x* will $g(x) \leq g^{-1}(x)$. (3)

[10]

Possible solutions

2.1	$0=-2x+3$ $2x=3$ $x=\frac{3}{2}$ $\therefore \mathrm{T}\left(\frac{3}{2};0\right)$ T is on x-axis, let $y=0$	✓ $0=-2x+3$ ✓ $\therefore \mathrm{T}\left(\frac{3}{2};0\right)$ (2)
2.2	$y=-2(-1)+3$ $=5$ $y=-2(4)+3$ $=-5$ $\therefore x\in[-5;5]$ or $-5\le x\le 5$ Domain of g^{-1} is the range of g. Calculate y.	✓ interval ✓ notation (2)
2.3		✓ intercept ✓ end points ✓ shape (3)
2.3	$-2x+3=-\frac{1}{2}x+\frac{3}{2}$ $-2x+\frac{1}{2}x=-3+\frac{3}{2}$ $-\frac{3}{2}x=-\frac{3}{2}$ $x=1$ $\therefore y=-2(1)+3=1$ Point of intersection is (1 ; 1) $\therefore x\ge 1$	✓ equating ✓ *x* value ✓ $x\ge 1$ (3)

Worked-out example 3

Given: $f(x) = 2x^2$

3.1 Determine the equation of the inverse of *f* in the form $f^{-1}(x) = ...$ (2)

3.2 On the same set of *x*-axis, sketch the graphs of *f* and f^{-1}. Indicate clearly the intercepts with the axis, as well as another point on the graph of each f and f^{-1}. (4)

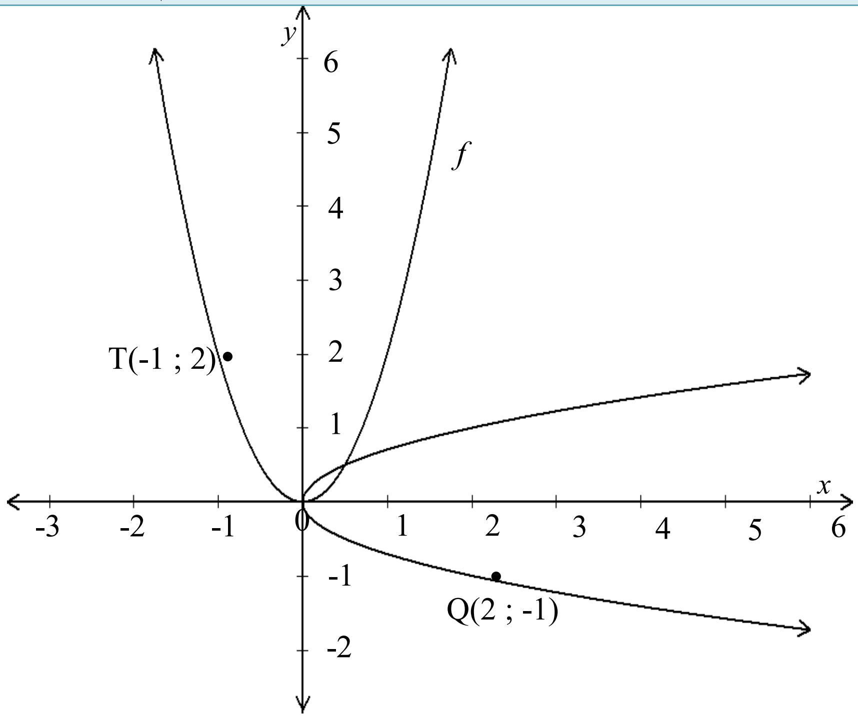
3.3 Determine the domain and range of both *f* and f^{-1}. (4)

3.4 Is f^{-1} a function? Give a reason for your answer. (2)

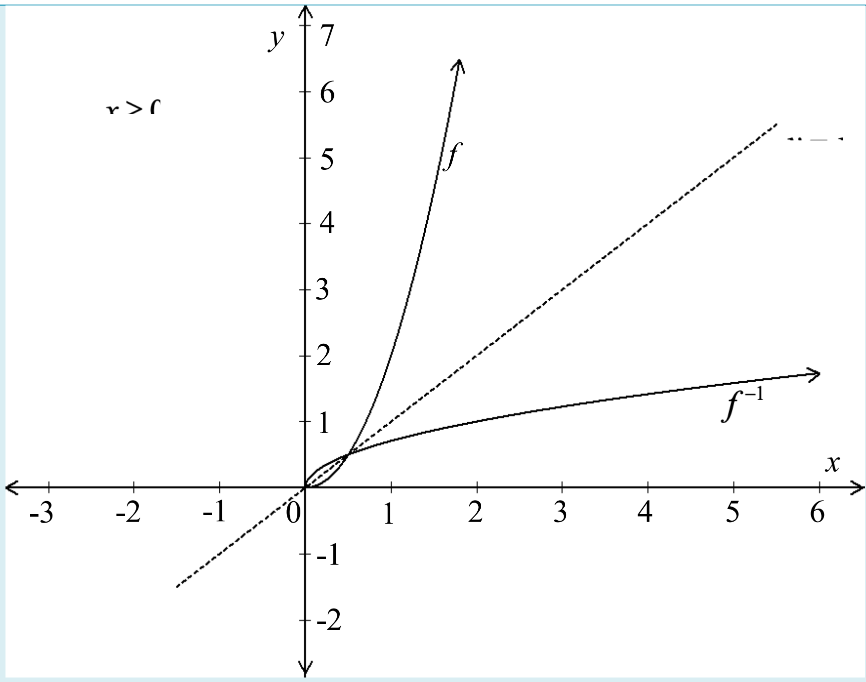
3.5 How can *f* be restricted so that the inverse is a function? (2)

3.6 On the separate set of axis, sketch both restricted function and their inverses. (4)

Possible solution

3.1	$y=2x^2$ $x=2y^2$ $\frac{1}{2}x=y^2$ $\therefore y=\pm\sqrt{\frac{1}{2}x}$ $f^{-1}(x)=\pm\sqrt{\frac{1}{2}x}$	✓ swapped *x* and *y* ✓ equation (2)
3.2		*f* ✓ one point ✓ shape f^{-1} ✓ one point ✓ shape (4)
3.3	For *f*: Domain: $x\in\mathbb{R}$ or $x\in(-\infty;\infty)$ Range: $y\in[0;\infty)$ For f^{-1}: Domain: $x\in[0;\infty)$ Range: $y\in\mathbb{R}$ or $y\in(-\infty;\infty)$	✓ $x\in\mathbb{R}$ ✓ $y\in[0;\infty)$ f^{-1} ✓ $x\in[0;\infty)$ ✓ $y\in\mathbb{R}$ (4)
3.3	No, One value of *x* is associated with two values of *y*.	✓ No ✓ reason (2)
3.4	$x\leq 0$ and $x\geq 0$	✓ $x\leq 0$ ✓ $x\geq 0$ (2)

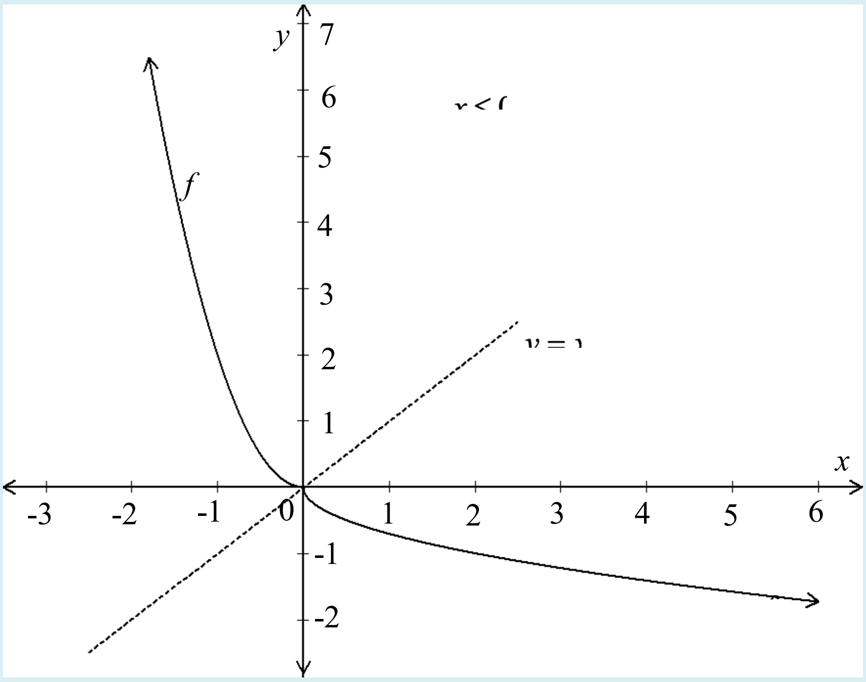
3.5



✓ shape of f

✓ shape of f^{-1}

And



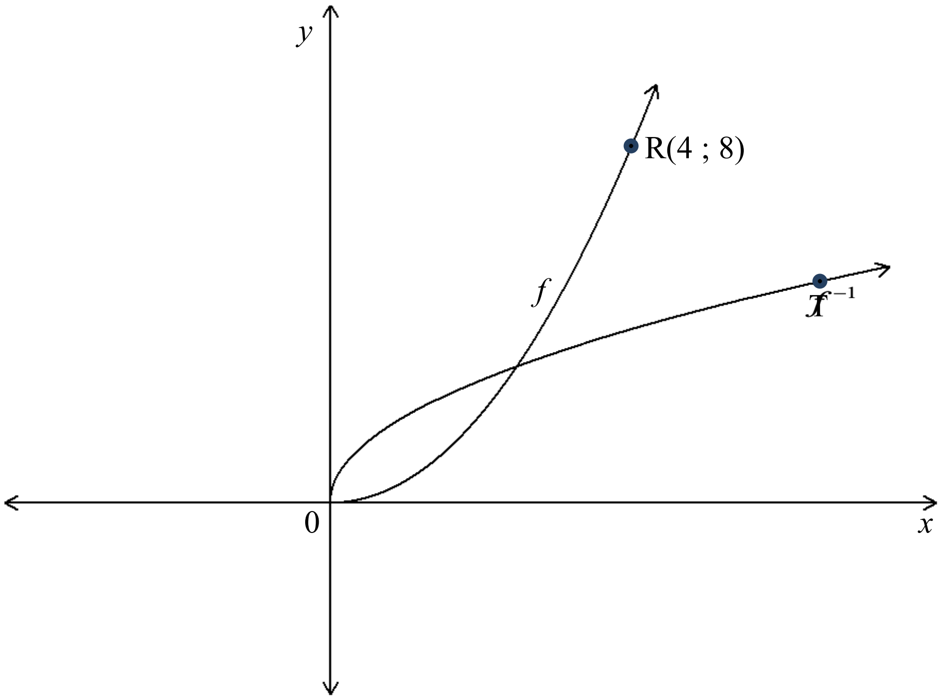
✓ shape of f

✓ shape of f^{-1}

(4)

Worked-out example 4

In the diagram below, the graph of $f(x) = ax^2$ is drawn in the interval $x \geq 0$.
The graph of the inverse, f^{-1} is also drawn. R(4 ; 8) is a point on *f* and T is on f^{-1}.



4.1	Is f^{-1} a function? Motivate your answer.	(2)
4.2	Determine the equation of *f*.	(2)
4.3.	If T is the reflection of R in the line $y = x$, write down the coordinates of T.	(1)
4.4	Write down the equation of f^{-1} in the form $y = ...$	(3)

Possible solution

4.1	Yes One value of *x* is associated with one value of *y*	✓ Yes ✓ Reason (2)
4.2	$8 = a(4)^2$ $8 = 16a$ $a = \frac{1}{2}$	✓ subst. (4;8) ✓ $a = \frac{1}{2}$ (2)
4.3	$\mathrm{T}(8;4)$	✓ answer (1)
4.4	$y = \frac{1}{2}x^2$ $x = \frac{1}{2}y^2$ $y = \pm\sqrt{2x}$	✓ Swapped *x* and *y* ✓ $y = \pm\sqrt{2x}$ (2)

How to determine the inverse:

$y = b^x$

$x = b^y$

$y = \log_b x$

Exponential function

Standard for: $f(x) = ab^x$

Inverse: $f^{-1}(x) = \log_b x$

A logarithm is a way of writing an exponential equation with the exponent as the subject of the equation.

Definition: In the equation $x = b^y$ where b > 0 and $b \neq 1$, b is referred to as the base and y is the exponent.

In general, converting from exponential form to logarithmic form is as follows:

$$x = b^y \text{ then } y = \log_b x \text{ where } b > 0 \text{ and } b \neq 1 \text{ and } x > 0$$

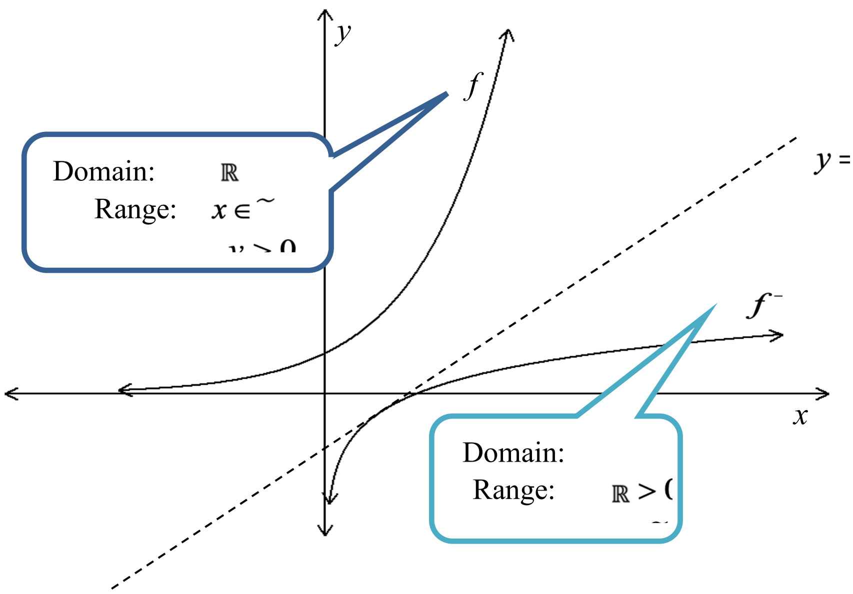
The logarithmic function is the inverse of the exponential.

CHANGE FROM LOG TO EXPONENTIAL FORM

1. $y = \log_b x$

   $b^y = x$

Shape



Worked-out example 5

Given: $g(x) = 3^x$

5.1 Write down the range of *g*. (1)

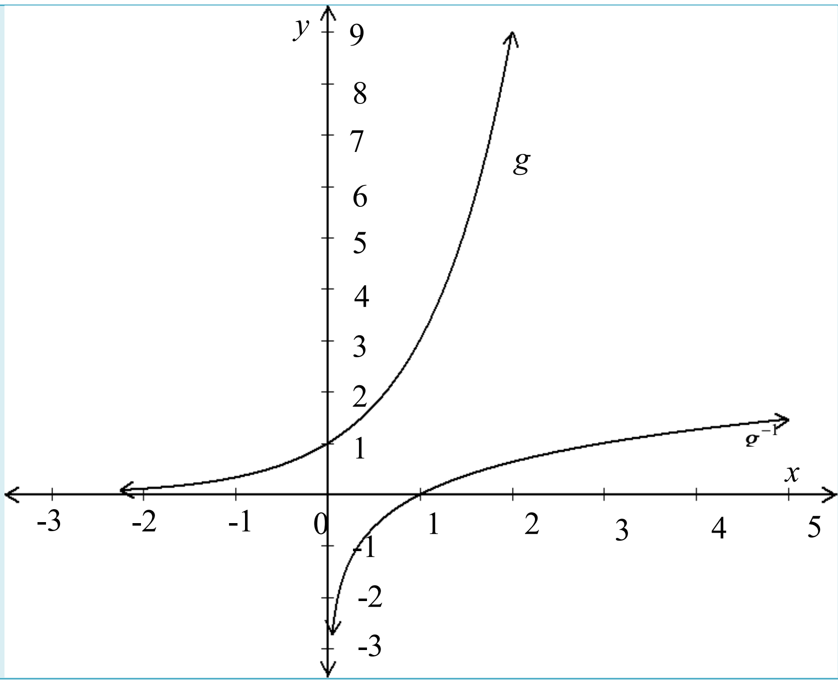
5.2 Determine the equation of g^{-1} in the form $y = ...$ (2)

5.3 Sketch the graph of g and g^{-1} on the same set of axis. (4)

5.4 Write down the domain of g^{-1}. (1)

5.5 Determine the equation of the asymptote of $r(x) = g(x-2)$. (2)

Possible solution

5.1	$y \in (0;\infty)$ $y \in {}^{\sim}; y > 0$	✓✓ $y \in (0;\infty)$ (2)
5.2	$y = 3^x$ $x = 3^y$ $y = \log_3 x$	✓ swapped *x* & *y* ✓ equation (2)
5.3		✓ intercept ✓ shape ✓ intercept ✓ shape (4)
5.4	$x \in (0;\infty)$	✓ $x \in (0;\infty)$ (1)
5.5	$x = 2$	✓✓ $x = 2$ (2)

Worked-out example 6

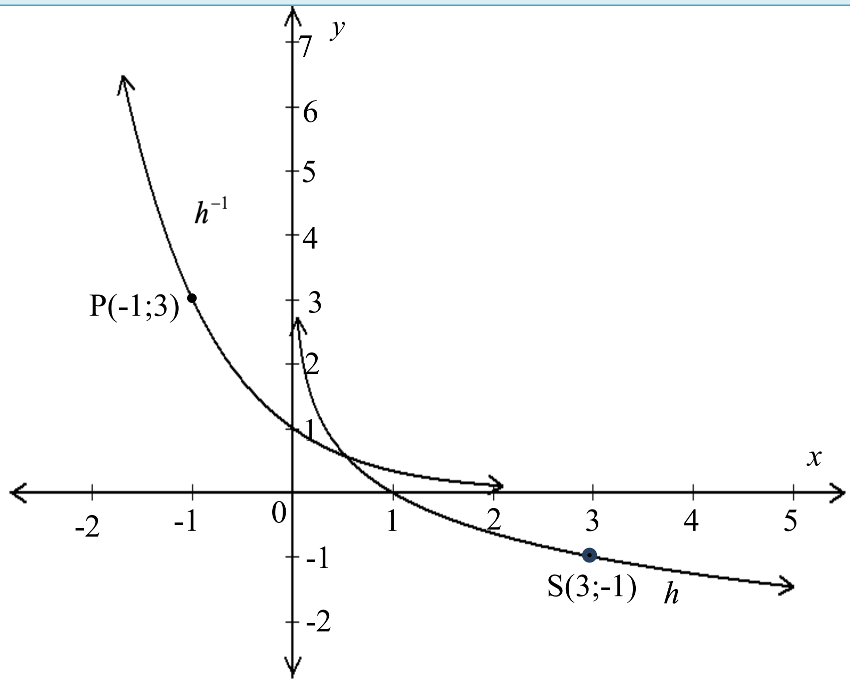
Given the function: $h(x) = \log_{\frac{1}{2}} x$.

6.1 Show that the inverse can be express as $h^{-1}(x) = 2^{-x}$. (2)

6.2 Sketch both h and h^{-1} on the same set of axis, clearly indicate all intercepts with axis and one point on each graph. (4)

6.3 Determine the domain of $f(x) = h(x+3)$. (1)

Possible solution

<table>
<tr>
<td>6.1</td>
<td>$y = \log_{\frac{1}{3}} x$
$x = \log_{\frac{1}{3}} x$
$y = \left(\frac{1}{3}\right)^x$
$y = 3^{-x}$</td>
<td>✓ swapped x & y
✓ equation
(2)</td>
</tr>
<tr>
<td>6.2</td>
<td>
</td>
<td>h
✓ one point
✓ shape
h^{-1}
✓ one point
✓ shape
(4)</td>
</tr>
<tr>
<td>6.3</td>
<td>$x \in (-2; \infty)$</td>
<td>✓ $x \in (-2; \infty)$
(3)</td>
</tr>
<tr>
<td colspan="3"></td>
</tr>
</table>

3.2 Practice Questions

1. Given: $f(x) = 2x^2 - 8$

1.1 Calculate the *x*-intercepts of *f*. (2)

1.2 Calculate the *y*-intercept of *f*. (2)

1.3 Write down the coordinates of the turning point of *f*. (2)

1.4 Sketch the graph of *f*, clearly show all intercepts and turning point (3)

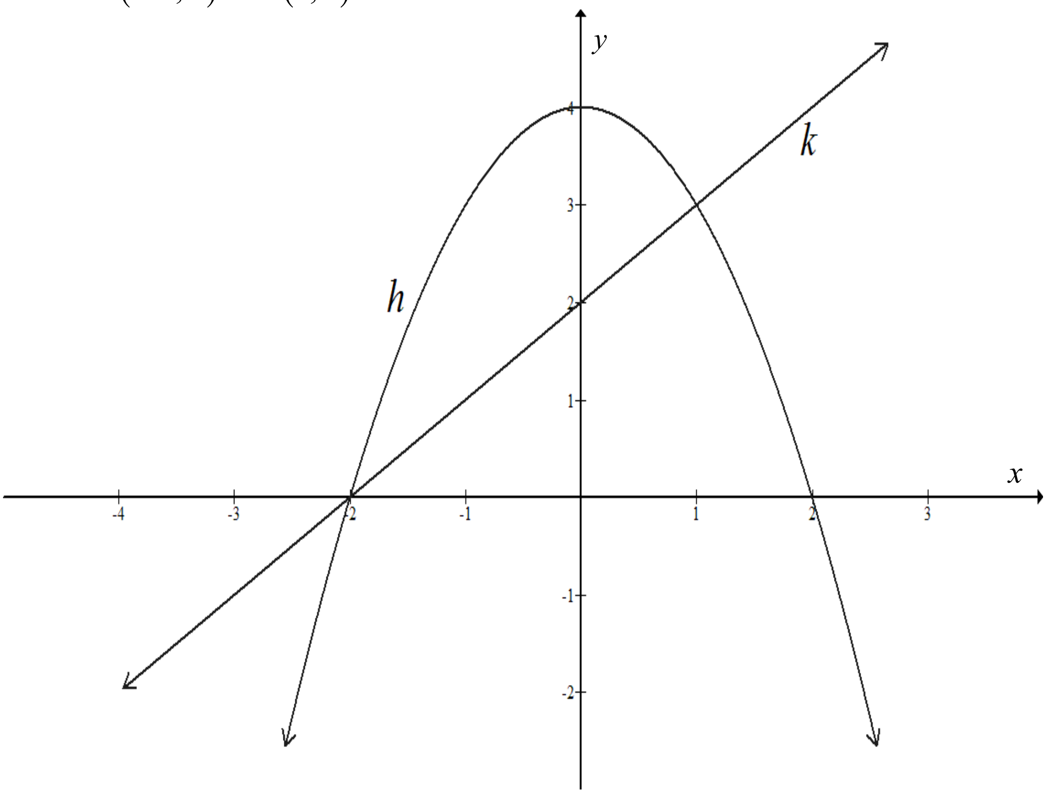
1.5 Determine the range of *f(x)*. (2)

1.6 For which values of *x* will $f(x) \leq 0$ (2)

1.7 Determine w, if $(w;10)$ is a point on the graph of *f (x)*. (2)

[15]

2. The sketch below shows the graphs of $h(x) = ax^2 + q$ and $k(x) = x + 2$. The graphs intersect at $(-2\,;0)$ and $(1\,;3)$.



2.1 Determine the values of *a* and *q*. (3)

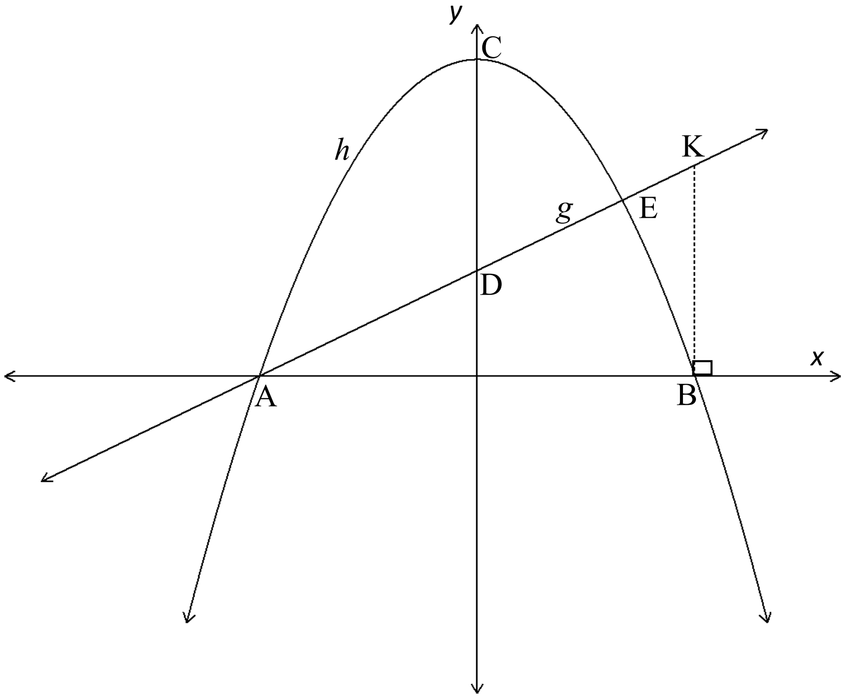
2.2 For which value(s) of *x* is $h(x) \geq k(x)$? (2)

2.4 Determine the interval where *h* decreases. (2)

2.5 Write down the range of *p* where $p(x) = -h(x) + 2$. (4)

[13]

3 The graph of $h(x) = -x^2 + 9$ and $g(x) = x - 3$ are sketched below. A and B are the *x*-intercepts of *f*. C and D are the y-intercepts of h and *w* respectively. K is a point on w such that BK// x-axis. H and w intersect at A and E.



3.1 Write down the coordinates of C. (1)

3.2 Write down the coordinates of D. (1)

3.3 Determine the length of CD. (1)

3.4 Calculate the coordinates of B. (3)

3.5 Determine the coordinates of E, a point of intersection of *f* and *g*. (4)

3.6 For which values of x will:

3.6.1 $h(x) < g(x)$ (2)

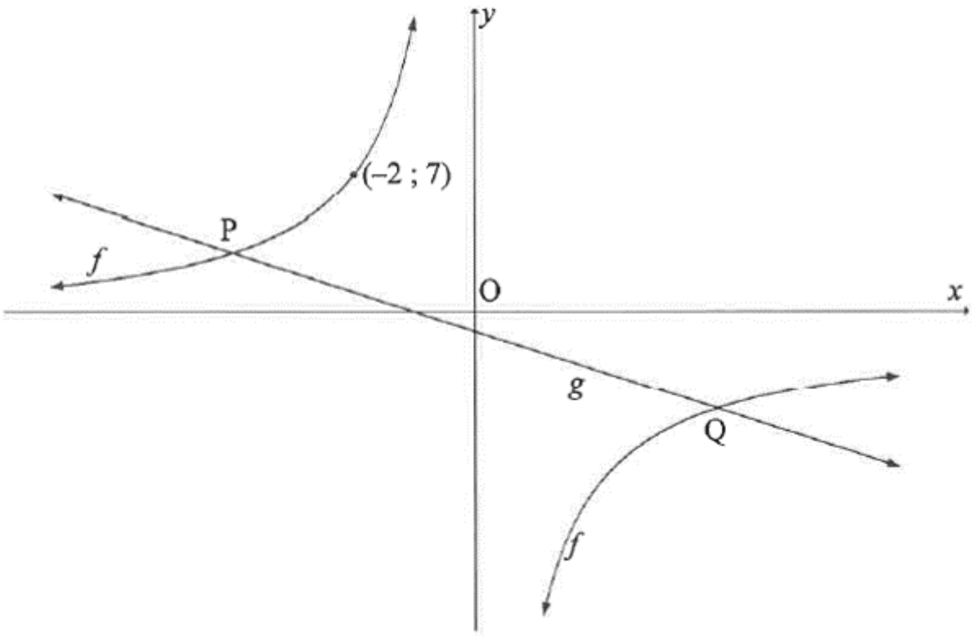
3.6.2 $h(x).g(x) \geq 0$ (2)

3.7 Calculate the length of AK. (4)

4 Given: $p(x) = \dfrac{4}{x} - 2$

4.1 Determine the equation of the asymptote of *p*. (1)
4.2 Calculate the x-intercept of *p*. (2)
4.3 Sketch the graph of p and clearly show the all the intercepts with the axis. (3)
4.4 Write down the domain of *p*. (2)
4.5 Calculate the value of d if the line of symmetry is $y = -x + q$ (1)

5 The sketch below shows the graph of $f(x)=\frac{a}{x}+q$. Point W(-2 ; 7) lies on f.

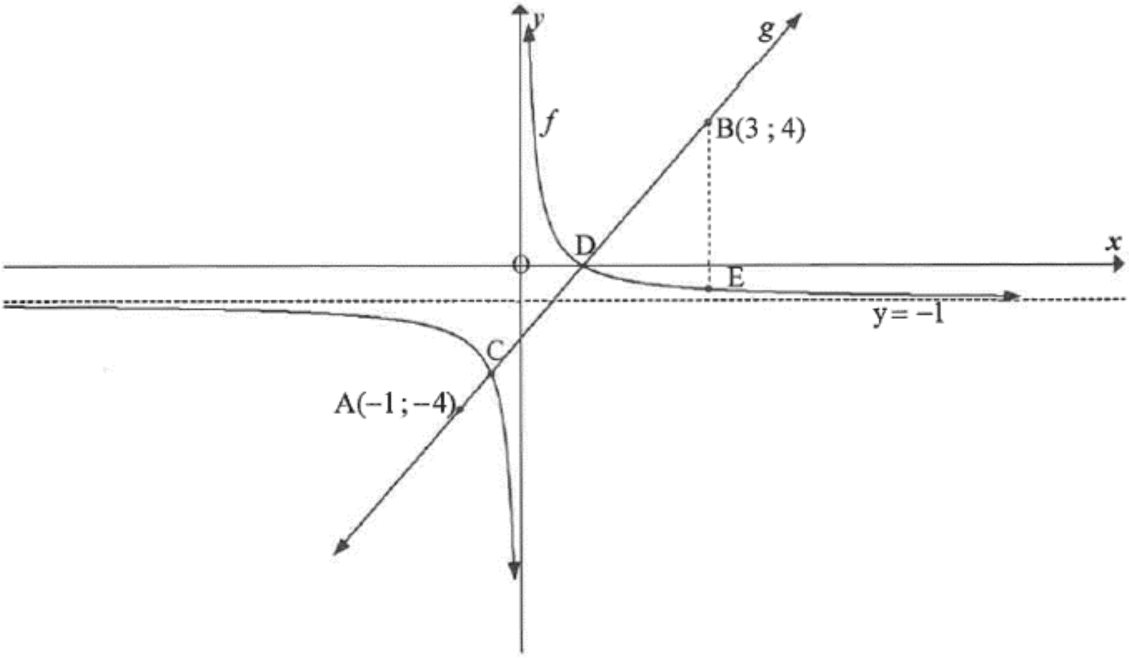


5.1	Write down the value of *a* and *q*.	(3)
5.2	Determine range of *f*.	(2)
5.3	Calculate the equation of axis of symmetry with negative gradient.	(2)
5.4	Calculate the *x*-coordinate of the points of the intersection of *f* and line of symmetry.	(4)

6 The sketch below shows f and g, the graphs of $g(x) = \frac{3}{x} + 2$ and $k(x) = ax + q$

Respectively. Points A(-1 ; -4) and B(3 ; 4) lie on the graph *g*.

The two graphs intersect at the points C and D



6.1 Show that $a = 2$ and $q = -2$ (2)

6.2 Determine the values of x for which $f(x) = g(x)$. (4)

6.3 For which values of x is $f(x) < g(x)$ (2)

6.4 Write down the length of h if $h(x) = f(x) + 4$ (1)

7 Given: $f(x) = -3^x + 9$

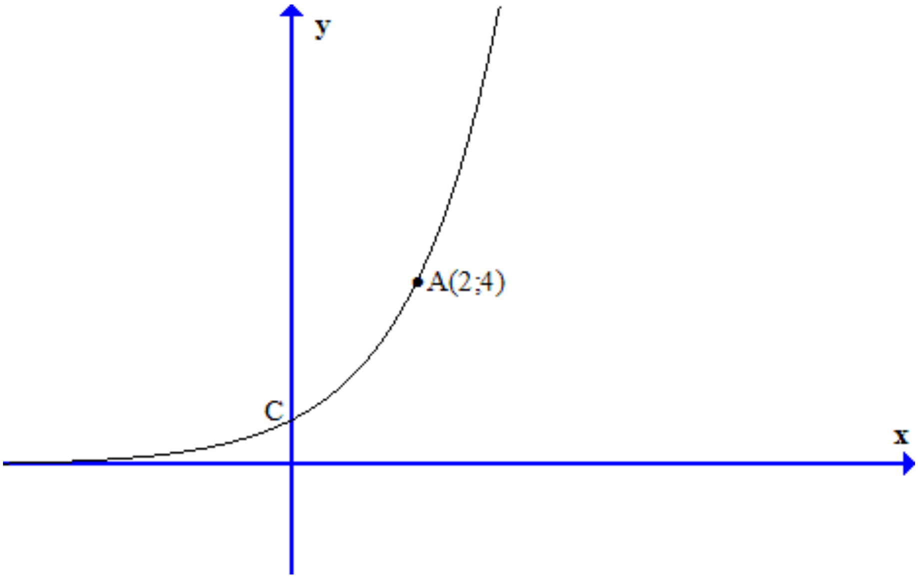
7.1 Write down the equation of the asymptote of *f*. (2)

7.2 Determine the x- and y-intercepts of *f*. (3)

7.3 Draw sketch graph of *f*. Indicate the intercepts with the axes and the asymptote. (3)

7.4 Write down the equation of *g(x)*, if $h(x) = -f(x)$. (2)

8 The graph of $f(x) = a^x$ is represented in the diagram below. A(2 ; 4) is a point On the graph.



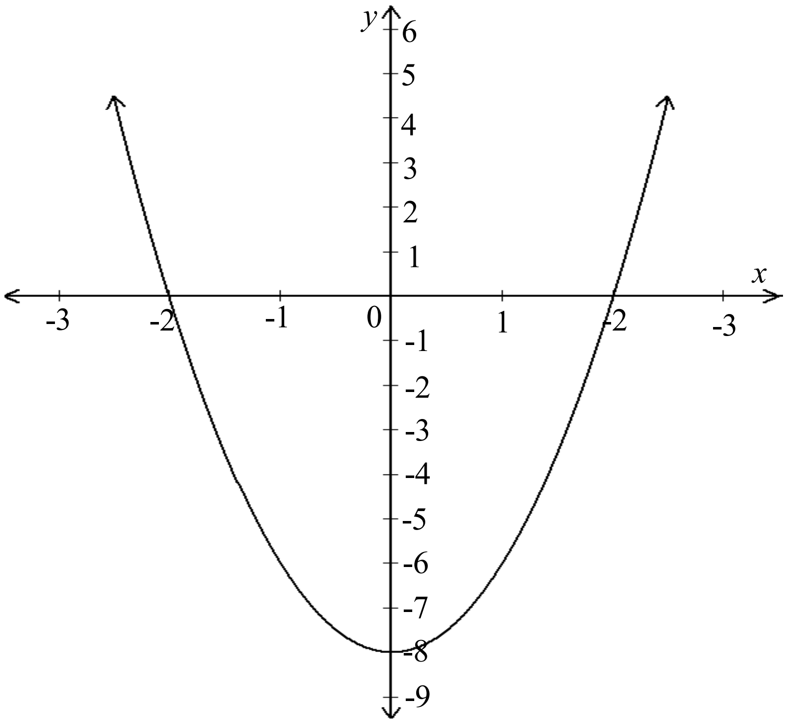
8.1 Determine the equation of *f*. (3)

8.2 Write down the coordinates of C, y-intercept of *f*. (2)

8.3 Write down the range of *f*. (1)

8.3 Determine the equation of *k*, if *f* is reflected about the *y*-axis and moved Two downwards. (2)

3.3 POSSIBLE SOLUTION

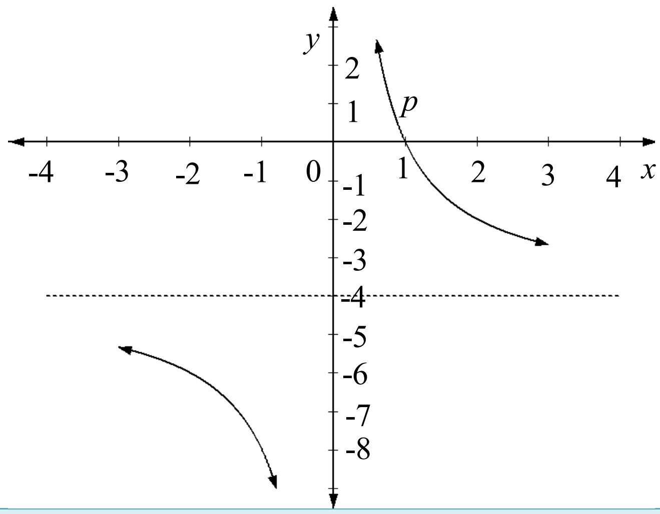
1.1	For x-intercept: le $y=0$ $0=2x^2-8$ $0=x^2-4$ $0=(x-2)(x+2)$ $x=2$ or $x=-2$	✓ $y=0$ ✓ factors ✓ answers (3)
1.2	$y=2(0)^2-8$ $y=-8$	✓ $x=0$ ✓ answer (2)
1.3	$(0;-8)$	✓✓ $(0;-8)$ (2)
1.4		✓ x- & y-interepts ✓ turning point ✓ shape (3)
1.5	$y\in[-8;\infty)$ or	✓ interval ✓ notation (2)
1.6	$x\in[-2;2]$ or $-2\le x\le 2$	✓ interval ✓ notation (2) ✓ interval ✓ notation (2)

1.7	$10 = 2w^2 - 8$ $18 = 2w^2$ $9 = w^2$ $w = \pm\sqrt{9}$ $w = \pm 3$	✓ subst. ((w ; 10) ✓ answers
		[15]

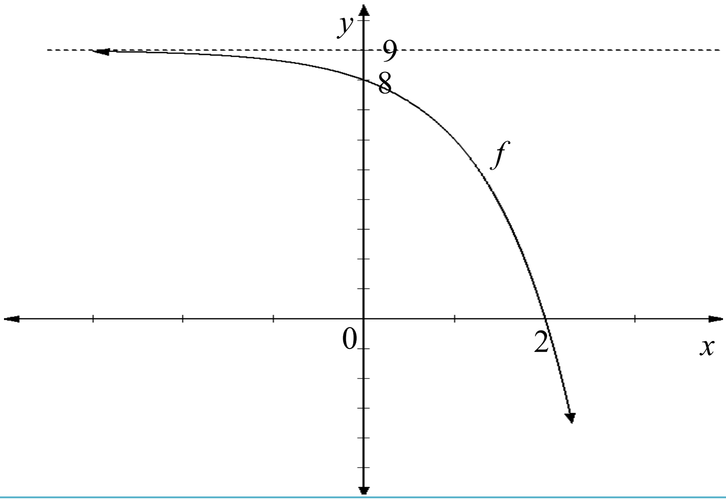
2.1	$q = 4$ (y-intercept or turning point) $0 = a(-2)^2 + 4$ $-4 = 4a$ $\therefore a = -1$	✓ $q = 4$ ✓ subst. (-2 ; 0) ✓ value of a (3)
2.2	$x \in [-2;3]$ or $-2 \leq x \leq 2$	✓ interval ✓ notation (2)
2.3	$x \in (0;\infty)$	✓ interval ✓ notation (2)
2.4	$p(x) = -(-x^2 + 4) + 2$ $\quad = x^2 - 4$ $\therefore y \in [-4;\infty)$	✓ $p(x) = x^2 - 4$ ✓ interval ✓ notation (3)
		[10]

3.1	$(0;9)$ y-intercept of h	✓ $(0;9)$ (1)
3.2	$(0;3)$ y-intercept of h	✓ $(0;3)$ (1)
3.3	CD = 9 - 3 $\quad$ = 6 units	✓ 6 units (1)
3.4	$0 = -x^2 + 9$ $x^2 - 9 = 0$ $(x-3)(x+3) = 0$ $x = 3$ or $x = -3$	✓ $y = 0$ ✓ factors ✓ both answers (3)
3.5	$-x^2 + 9 = x + 3$ $0 = x^2 + x - 6$ $0 = (x+3)(x-2)$ $\therefore x = -3$ or $x = 2$	✓ equating h & g ✓ standard form ✓ factors ✓ answers

	$y = 2 + 3 = 5$ $\therefore \text{E}(2;5)$	✓ E(2 ; 5) (5)
3.6.1	$x \in (-3;2)$	✓ answer (1)
3.6.2	$x \in [0;3]$	✓ answer (1)
3.7	$y = 3 + 3$ $= 6$ $\therefore \text{BK} = 6$ units	✓ y value at K ✓ Length of BK (2)
		[15]

4.1	$x = 0$ $y = -2$	✓ $x = 0$ ✓ $y = -2$ (2)
4.2	$0 = \frac{4}{x} - 2$ $2 = \frac{4}{x}$ $x = 2$	✓ $y = 0$ ✓ $x = 2$ (2)
4.3	 	✓ x-intercepts ✓ asymptotes ✓ shape (3)
4.4	$q = -4$	✓ $q = -4$ (1)
		[15]

6.1	$m = \dfrac{4-(-4)}{3-(-1)}$ $\quad = 2$ $a = 2$ $4 = 2(3) + q$ $q = -2$	✓ substitution ✓ $a = 2$ ✓ subst.(3;4) or (-1;-4) ✓ $q = -2$ (4)
6.2	$0 = \dfrac{3}{x} - 2$ $2 = \dfrac{3}{x}$ $x = \dfrac{3}{2}$	✓ $y = 0$ ✓ answer (2)
6.3	$x \in \left(\dfrac{3}{2};0\right)$	✓✓ answer (2)
6.4	$h(x) = \dfrac{3}{x} - 2 + 4$ $\quad = \dfrac{3}{x} + 2$ $y \in {}^{\sim}, y \neq 2$	✓ answer (1)
		[09]

7.1	$y=9$	✓ $y=9$ (1)
7.2	$y=-3^0+9$ $=-1+9$ $=8$ $0=-3^x+9$ $3^x=9$ $3^x=3^2$ $\therefore x=2$	✓ $y=0$ ✓ answer ✓ $x=0$ ✓ answer (4)
7.3		✓ x- & y-intercepts ✓ asymptote ✓ shape (3)
7.4	$h(x)=-(-3^x+9)$ $=3^x-9$	✓✓ answer (2)
		[10]

4. Financial mathematics

4.1 Notes/Summaries/Key Concepts

TOPICS	KEY WORDS	HOW TO LEARN	FORMULA
COMPOUND INTEREST/ **INFLATION**	Appreciation/ escalates/ increases (To gain more value)	The account holder deposits money and stay away from the bank for a considerable time and only comes back to collect the money at the end of period agreed upon. Remember the interest is not the same every year.	$A = P(1+i)^n$ **A** is the final amount **P** is the principal /sum of money invested or borrowed ***n*** is the number of periods (How long/how many etc..)

Worked-out example

1. Andrew invest R1200 at an interest rate of 10% compounded quarterly. Calculate how much will Andrew's investment be worth after 5 years.

Possible solution

1.	$A = ?$ $P = 1200 \quad n = 5\,years \times 4 = 20$ $i = \frac{0.1}{4}$ $A = P(1+i)^n$ $= 1200\left(1+\frac{0.1}{4}\right)^{20}$ $R1\,966{,}34$	✓correct n ✓substitution ✓ answer
		(3)

2. R15 000 is invested into a savings account. Calculate the value of the investment of the savings after 6 years if interest rates is 13% compounded monthly.

Solution

2.	$n = 6 \times 12 = 72$ $A = ?$ $P = 15000$ $i = \frac{0.13}{12}$ $A = P(1+i)^n$ $= 15000\left(1+\frac{0.13}{12}\right)^{72}$ R32585.11	✓$i = \frac{0.13}{12}$ ✓substitution ✓ answer (3)

3. Two friends each receive an amount of R8 000 to invest for a period of 6 years. They invest the money as follows:

- Redishe: 8,5% per annum simple interest. At the end of 6 years, Redishe will receive a bonus of exactly of 6% of the principal amount.
- Thato: 8% per annum compounded quarterly.

Who will have the bigger investment after 6 years? Justify your answer appropriate calculations. (6)

Possible solution

3.	**Redishe:** $A = P(1+in)$ $= 8000(1+6\times 0.085)$ **=** $R12080$ $6\% \times R8000$ **=** $R480$ $\text{Total} = R8000 + R480$ $= R8480$ **Thato:** $i = 0.08 \div 4 = 0.02$ $n = 6 \times 4 = 24$ **months**	✓substitution ✓answer ✓ $R480$ $= R8480$

	$A = P(1+i)^n$ $= 8000(1+0.02)^{24}$ $= R12867.50$ $\therefore$ **Thato will have the bigger investment**	✓substitution ✓ answer (6)

TOPICS	KEY WORDS	HOW TO LEARN	FORMULA
Depreciation on a reducing balance	Scrap value /old	The amount decreasing is not the same year on year.	$A = P(1-i)^n$

Worked-out example

1. The value of motor cycle depreciates by 12% per year on the reducing balance method. The motor cycle is currently worth R45 000.Calculate the value of the motor cycle after five years.

Solution

1.	$A = ?$ $P = 45\,000$ $i = \frac{12}{100} = 0.12$ $A = P(1-i)^n$ $= 45000(1-0.12)^5$ $= R23\,747.94$	✓correct formula ✓substitution ✓ answer
		(3)

TOPICS	KEY WORDS	HOW TO LEARN	FORMULA
Simple Interest/Hire purchase/Straight line basis	Straight line/linear	Interest stays the same during the investment period. Hire purchase simple means increasing **Straight line basis –it is simple mean decrease/depreciate**	$A = P(1-in)$ $A = P(1-in)$

Worked-out example

1. A car worth R150 000 depreciates at a rate of 9% (simple interest) p.a. How much the car worth after 5 years.

Solution

1.	$A = ?$ $P = 150\,000$ $i = 0.09$ $A = P(1 - in)$ $= 150000(1 - 0.09 \times 5)$ $= R82\,500$ **The car will be worth R82 500 after 5 years**	✓correct formula ✓substitution ✓ answer
		(3)

2. The value of a piece of machinery depreciates from $R20000$ to $R10000$ in years. What is the rate of depreciation, correct to two decimal places, if is calculated on the straight line method. (4)

Solutions

2.	$A = 10\,000$ $P = 20\,000$ $n = 4$ $i = ?$ $A = P(1 - in)$ $10000 = 20000(1 - 4i)$ $\frac{10000}{20000} = \frac{20000(1-4i)}{20000}$ $\frac{1}{2} - 1 = -4i$ $-0.5 = -4i$	✓formula ✓correct substitution

	$i = 0.125$ $\therefore i = 0.125 \times 100$ $r = 12.5\%$	✓ $i = 0.125$ ✓ $r = 12.5\%$
		(4)

TOPICS	KEY WORDS	HOW TO LEARN	FORMULA
Nominal and effective interest rate	Nominal is compounded *m* times in a year. Effective is compounded annually.	Different compounding periods: • The main issue is compounding periods. • Interpret the concepts on how to deal with compounding periods: ○ Half yearly / Bi-nnually/ Semi-annually: $n \times 2$ and $\frac{i}{2}$ ○ Yearly: n and i never change ○ Quarterly: $n \times 4$ and $\frac{i}{4}$ ○ Monthly: $n \times 12$ and $\frac{i}{12}$ ○ Daily: $n \times 365$ and $\frac{i}{365}$ ○ Weekly: $n \times 52$ and $\frac{1}{52}$	$1 + i_{eff} = \left(1 + \frac{i^m}{m}\right)^m$ From the formula: • i^m is for the nominal interest rate and • i_{eff} is for the effective interest rate.

Worked-out example

1. Convert a nominal interest rate of 7% per annum compounded semi-annually to the effective annual interest rate.

(4)

Solution

2.	$1+i_{eff}=\left(1+\frac{i^m}{m}\right)^m$ $i_{eff}=\left(1+\frac{0.07}{2}\right)^2-1$ $i=0.071225$ $\therefore i=0.071225\times100$ $\therefore i=7.12\%$ **is the effective annual interest rate.**	✓correct formula ✓making i the subject of the formula ✓ $i=0.071225\times100$ ✓ final answer
		(4)

TOPICS	KEY WORDS	HOW TO LEARN	FORMULA

Using logarithm to determine "n" (number of period(s))	$2^3 = 7$ • 3 is the exponent • 2 is a base • 7 is a number	To change exponential form to logarithm or Logarithm to exponential	$2^3 = 7$ Introduce law of changing a base by introducing Log both sides: e.g. $\log 2^3 = \log 7$ $3\log 2 = \log 7$ $3 = \dfrac{\log 7}{\log 2}$ $3 = \log_2 7$ • Write as single log • Manipulate an exponent and a number but, • Base of exponent is also a base for logarithm. eg. $2^3 = 7$ $\log_2 7 = 3$

Worked-out example

1. John deposits R5 000 into a savings account. The interest paid is 8% p.a., compounded monthly. How long will it take her to double his savings?

2. Lerato repays a loan of R300 000 by means of equal monthly instalments of R7 000 per months starting one month after the loan was drawn. The bank charges interest on the outstanding balance of the loan at 13% p.a., compounded monthly. Calculate how many monthly payments it will take to repay the loan.

Solutions

1.	$P = 5000 \quad n = ? \quad i = 0.08 \quad A = 10000$ $A = P(1+i)^n$ $10000 = 5000\left(1 + \dfrac{0.08}{12}\right)^n$ $\dfrac{10000}{5000} = \left(1 + \dfrac{0.08}{12}\right)^n$	 ✓correct formula ✓substitution

	$2=\left(1+\frac{0.08}{12}\right)^{n}$ $n=\log_{\left(1+\frac{0.08}{12}\right)}2$ $n=104.32$ **months**	✓ introduction of logs ✓ answer (4)
2.	$n=?\quad i=0.13\quad P_V=300000\quad x=7000$ $Pv=\frac{x\left[1-(1+i)^{-n}\right]}{i}$ $300000=\frac{7000\left[1-\left(1+\frac{0.13}{12}\right)^{-n}\right]}{\frac{0.13}{12}}$ $\left(1+\frac{0.13}{12}\right)^{-n}=1-4.64285714286$ $\left(1+\frac{0.13}{12}\right)^{-n}=-3.64285714286$ $n=\log_{\left(1+\frac{0.13}{12}\right)}3.64285714286$ $=119.98$ **months**	✓ formula ✓ substitution ✓ calculation ✓ introduction of logs ✓answer (5)

TOPICS	KEYWORDS	HOW TO LEARN	FORMULA
Future value of annuities	• Savings • Investing • Retirement annuities • Deposits are made	It is very important to investigate all the options when investing money.	$Fv=\frac{x\left[(1+i)^{n}-1\right]}{i}$ $Fv\rightarrow$ future value $x\rightarrow$ value of the instalment $i\rightarrow$ interest rate $n\rightarrow$ number of payments (deposits)/number of periods

Worked-out examples 1

Ricky opened a savings account with a single deposit of R2 000 on first April 2018. He then makes 18 monthly deposits of R800 at the end of every month. His first payment is made on 30 April 2018 and his last payment on 30 September 2019. The account earns interest at 16% per annum compounded monthly.

Determine the amount that should in his account immediately after his last deposit is made (that is September 2019).

Possible solution

1.	$A = P(1+i)^n$ $= 2000\left(1+\frac{0.16}{12}\right)^{18}$ $A = 2538.47$ $Fv = \frac{x\left[(1+i)^n - 1\right]}{i}$ $Fv = \frac{800\left[\left(1+\frac{0.16}{12}\right)^{18} - 1\right]}{\frac{0.16}{12}}$ $= R16154.08$	✓ substitution ✓answer ✓correct formula ✓✓correct substitution ✓ answer
		(6)

Worked out example

Jack's took out an ordinary annuity for him when he was born. The monthly payment is R200 and the interest rate is 10% p.a., compounded monthly.

Peter's parents started an annuity fund for him when he turned 12 years old. The monthly payment is R800 and the interest is 12% p.a., compounded monthly.

Both Jack and Peter will receive the money from the annuities on their 21^{st} birthday.

a)	Who will receive more money on their 21^{st} birthday?	(6)
b)	How much money did each set of parents contribute in total?	(2)
c)	Express the interest earned as a percentage of the total contribution in each case.	(4)
d)	What do you conclude from your answer to a) to d)?	(3)

Possible Solution

a.	Jack: $Fv = \frac{x\left[(1+i)^n - 1\right]}{i}$ $Fv = \frac{200\left[\left(1+\frac{0.1}{12}\right)^{21\times12} - 1\right]}{\frac{0.1}{12}}$	✓✓ correct substitution ✓answer

	$= R170\ 290.05$ Peter: $Fv = \dfrac{800\left[\left(1+\dfrac{0.12}{12}\right)^{9\times12} -1\right]}{\dfrac{0.12}{12}}$ $= R154\,314.06$	✓✓ correct substitution ✓answer (6)
b.	Jack: parents' contribution $= R200\times21\times12 = R50\ 400$ Peter: parents' contribution $= R800\times9\times12 = RR86\ 400$	✓correct answer ✓correct answer (2)
c.	Jack: parents' interest $= R170\ 290.05\ - R50\,400$ $= R119\ 890.05$ Peter: parents' interest $= R154\ 314.06\ - R86\ 400$ $= R67\,914.06$	✓✓correct interest ✓✓correct interest (4)
d.	Jack : $\dfrac{R119\,989.05}{R50\,400}\times100\% = 237.88\%$ Peter: $\dfrac{R67\,914.6}{R86\,400}\times100\% = 78.60\%$ Jack: parents' contributed less than Peter parents' did, but Jack's annuity earned much more interest, because it ran for 21 years, as opposed to 9 years. The longer an annuity should therefore be viewed as a long term investment.	✓answer ✓answer ✓correct explanation (3)

TOPICS	KEY WORDS	HOW TO LEARN	FORMULA
Present value of annuities	• Loan • borrow • bond • mortgage	• P_v is the amount of money owed • x is the instalments depending on how it is compounded. • Deposit paid can be given as an amount (e.g R4 000) or in percentage (e.g 10% of the initial amount).	$Pv = \dfrac{x\left[1-(1+i)^{-n}\right]}{i}$

Worked out example

1. Themba takes out a loan of R100 000 to buy a house. The loan is repaid over 20 years. The interest paid on the amounts outstanding is 8.5% pa, compounded monthly. Calculate the monthly instalments over 20 years. (3)

2. A couple buys a car for R450 000. They put down a deposit and get a car for the balance at a rate of 11% p.a., compounded monthly. The couple calculates that it will take 6 years to repay the loan if they make a regular monthly payments of R3 800.
   a) How much money have they borrowed from the bank? Leave your answer to the nearest thousands rand. (2)
   b) Discuss their final payment. (2)
   c) Calculate deposit price of the car. (1)
   d) By the time they have paid for their car in full, how much interest will they have paid the bank? (2)
   e) If they increase their monthly payments by R300, how long will it take them to repay their loan? (3)
   f) If they decrease their monthly payments by R300, how long will it take them repay their loan? (3)

3. A loan of $R100\ 000$ is repaid over a period of 6 years, from the time that the loan is taken ,by equal monthly payments at an interest rate of 6.5% per annum compounded monthly. If payments start 5 months after the loan is granted, what monthly payments are required to repay the loan? (5)

Possible solution

1.	$Pv = R100\ 000;\ i = \frac{8{,}5\%}{12};\ n = 20\times 12 = 240$ $Pv = \dfrac{x\left[1-(1+i)^{-n}\right]}{i}$ $100\ 000 = \dfrac{x\left[1-\left(1+\frac{0.085}{12}\right)^{-240}\right]}{\frac{0.085}{12}}$ $x = R867.82$	✓ formula ✓ correct substitution ✓ making x the subject of the formula (3)
2.a)	$Pv = ?;\ i = \dfrac{0.11}{12};\ n = 6\times 12 = 72$	✓ $n = 6\times 12 = 72$

	$Pv=\frac{x\left[1-(1+i)^{-n}\right]}{i}$ $Pv=\frac{3800\left[1-\left(1+\frac{0.11}{12}\right)^{-72}\right]}{\frac{0.11}{12}}$ $=R199\,641.92$ **They will borrowed R199 000 from the bank.**	✓ substitution ✓ answer (3)
b)	Then final monthly payments will be slightly less than usual i.e $R3\,800-R641.92\ =3158.08$	✓✓ correct calculation (2)
c)	The deposit was $R450\ 000\ -R190\ 000$ $=R251\ 000$	✓ answer (1)
d)	The total payments $=72\times R3\ 800\ -R641.92$ $=R272\ 958.08$ They borrowed $R199\ 000$ from the bank, so the difference is interest $R272\ 958.08-R199\ 000\ \ =R73\ 958.08$	✓ correct calculation ✓answer (2)
e)	$Pv=\frac{x\left[1-(1+i)^{-n}\right]}{i}$ $199\ 000=\frac{5100\left[1-\left(1+\frac{0.11}{12}\right)^{-n}\right]}{\frac{0.11}{12}}$ $-\frac{3931}{6120}=-\left(1+\frac{0.11}{12}\right)$ $-n=\log_{\left(1+\frac{0.11}{12}\right)}\frac{3931}{6120}$ $-n=-4.241741904$ $n=4.241741904\div 12$ **4 years and 1 month.**	✓ substitution ✓ introduction of logs ✓answer (3)
f)	$199\ 000=\frac{3500\left[1-\left(1+\frac{0.11}{12}\right)^{-n}\right]}{\frac{0.11}{12}}$	✓ substitution

	$199\ 000 \times \frac{0.11}{12} = 3500\left(1+\frac{0.11}{12}\right)^{-n}$ $-\frac{2189}{4200} = -\left(1+\frac{0.11}{12}\right)^{-n}$ $-n = \log_{\left(1+\frac{0.11}{12}\right)} \frac{2189}{4200}$ $-n = -71.41329235$ $n = 71.41329235 \div 12$ **5 years and 11 months**	✓correct calculation ✓introducing of logs ✓answer in years and months (3)

3.	There are 4 months which is already elapsed $A = P(1+i)^n$ $= 100\ 000\ (1+\frac{0.065}{12})^4$ $= R123\ 491.5148$ **68 months payments to be made** $Pv = \frac{x\left[1-(1+i)^{-n}\right]}{i}$ $123\ 491.5148 = \frac{x\left[1-\left(1+\frac{0.065}{12}\right)^{-68}\right]}{\frac{0.065}{12}}$ $x = \frac{123\ 491.4948 \times \frac{0.065}{12}}{\left[1-\left(1+\frac{0.065}{12}\right)^{-68}\right]}$ $\therefore\ x = R\ 2175.85$ ✓ $\therefore$ The monthly payments would be $R2175.85$	✓substitution ✓ answer ✓correct formula ✓substitution ✓ answer (5)

TOPICS	KEY WORDS	HOW TO LEARN	FORMULA
Balance on loan/Outstanding balance/ loan settlement		TWO formulae to be used when calculating outstanding balance. • The present value of annuity or • the compound interest minus the future value of annuity.	$Pv = \frac{x\left[1-(1+i)^{-n}\right]}{i}$ Or $BL = A - FV$ $= P(1+i)^n - \frac{x\left[(1+i)^n - 1\right]}{i}$

Worked-out example

1. A loan of R250 000 is repaid over 5 years with equal monthly payments ,starting one month after the loan was granted.

1.1 Calculate the monthly repayments if the interest on the loan is 1% p.a., compounded monthly. (4)

1.2 The client has financial difficulties and makes only 19 payments. Calculate the balance of the loan at the end of the 19^{th} month. (4)

Possible solutions

1.1	$P_V = 250000 \quad x = ? \quad i = \frac{0.01}{12} \quad n = 5 \times 12$ $Pv = \dfrac{x\left[1-(1+i)^{-n}\right]}{i}$ $250000 = \dfrac{x\left[1-\left(1+\dfrac{0.01}{12}\right)^{-60}\right]}{\dfrac{0.01}{12}}$ $x = \dfrac{250000 \times \dfrac{0.01}{12}}{\left[1-\left(1+\dfrac{0.01}{12}\right)^{-60}\right]}$ $x = R4273.44$	✓formula ✓✓correct substitution ✓answer (4)

1.2	P_V = Balance of Loan $n = 60 - 19 = 41$ ✓ $i = \frac{0.01}{12}$ $x = R4273.44$ $Pv = \dfrac{4273.44\left[1-\left(1+\dfrac{0.01}{12}\right)^{-41}\right]}{\dfrac{0.01}{12}}$ $= R172181.14$	✓ $n = 60 - 19 = 41$ ✓answer ✓Correct substitution ✓answer (4)
	Or Balance of Loan $= A - F_V$ $= P(1+i)^n - \dfrac{x\left[(1+i)^n - 1\right]}{i}$ $= 250000\left(1+\dfrac{0.01}{12}\right)^{19} - \dfrac{4273.44\left[\left(1+\dfrac{0.01}{12}\right)^{19} - 1\right]}{\dfrac{0.01}{12}}$ $= R172181.14$	✓formula ✓✓ Correct substitution ✓ answer (4)

TOPICS	KEY WORDS	HOW TO LEARN	FORMULA
Sinking funds	• Escalate, • Increase, • Appreciation, • New valuable goods, then use (compound interests) • scrap value/old • depreciates • decrease on an reducing balance	Sinking Fund is an investment that is made to replace expensive equipment/ items in a few years' time. It is used as a "savings account" that will accumulate funds over a period of time, which will enable the investor to purchase expensive items or to fund expensive capital outlays in a few years' time (Saving in order to replace)	Formula New(appreciation) $A = P(1+i)^n$ $A = P(1-i)^n$ Sinking Fund = New – Old **or** Sinking Fund = Appreciation - Depreciation Calculate the monthly instalments using the future value of annuity formula. **Note that the calculated sinking fund becomes future value**

Worked-out examples

1. Lordwell buys new printers for R350 000.

1.1 How much will the printer be worth in 5 years' time if its value depreciates at 8% per annum on a reducing balance? (3)

1.2 After 5 years, the printer needs to be replaced. During this time , inflation remains constant at 6% per annum. Determine the cost of a new printer after 5 years. (3)

1.3 He plans to sell this printer at its book value and use the money towards a new printer. Calculate how much money he will need to put into a Sinking Fund to buy a new printer in 5year's time. (1)

1.4 Calculate the value of the monthly payments into the Sinking Fund if the interest is 9.5%p.a. c**ompounded monthly over the next 5 years.** (4)

Possible solutions

1.1	**Use compound depreciation with** $n=5 \quad i=0.08 \quad P=350000 \quad A=?$ $A=P(1-i)^n$ $=350000(1-0.08)^5$ $=R230678.5331$	✓formula ✓substitution ✓ answer (3)
1.2	**Use compound interest for inflation.** $n=5 \quad i=0.06 \quad P=350000 \quad A=?$ $A=P(1+i)^n$ $=350000(1+0.06)^5$ $=R468378.9522$	✓formula ✓ substitution ✓answer (3)
1.3	**Sinking fund=Cost of a new –book value of old** $=R468378.9522-R230678.5331$ $=R237700.42$	✓correct calculation ✓answer (2)
1.4	$F_v = 237700{,}42 \quad i=\frac{0.095}{12} \quad n=60 \quad x=?$ $Fv=\dfrac{x\left[(1+i)^n-1\right]}{i}$ 237700,42 = $\dfrac{x\left[\left(1+\dfrac{0.095}{12}\right)^{60}-1\right]}{\dfrac{0.095}{12}}$	✓ formula ✓ substitution ✓answer (3)

	$x = R3110.36$	

3.2 PRACTICE QUSTIONS A

3.2.1 Calculate how many years it will take an investment to double if it is invested at 8% per annum compounded semi-annually. (4)

3.2.2 Emely wants to purchase a car that costs 450 000. She is require to pay a 10% deposit and she will borrow the balance from a bank. Calculate the amount that Emely must borrow from the bank. (2)

3.2.3 The bank charges interest at 8% per annum, compounded monthly on the loan amount. Emely works out that the loan will carry an effective interest rate of 8,6% per annum. Is her calculation correct or not? Justify your answer with appropriate calculations. (4)

3.2.4 Emely takes out a loan from the bank for the balance of the purchased price and agrees to pay it back over 6 years. Her repayments start one month after her loan is granted. Determine her monthly instalments if interest is charged at 9.5% per annum compounded monthly. (4)

3.2.5 Emely can afford to repay R8 000 per month. How long will it take her the loan amount if she chooses to pay 8 000 as a repayment every month? (4)

3.3 PRACTICE QUESTIONS B

3.3.1 Khonani invested R12 000 in a bank. The investment remained in the bank for 12 years, earning interest at a rate of 8% per annum compounded annually. Calculate the amount at the end of 12years. (2)

3.3.2. Determine the final gain of Khonani's investment (1)

3.3.3 Show that her monthly instalment was R17 356,46 (4)

3.3.4 Calculate the outstanding balance on her loan at the end of the first year. (3)

3.3.5 Hence, calculate how much of the R4165550.40 that she paid during the first year, was taken by the finance company as payment towards the interest charged. (3)

3.4 PRACTICE QUESTIONS C

3.4.1 Mary invests R3 million into an account earning interest of 5% p.a. , compounded annually. How much will her investment be worth at the end of 4 years? (3)

3.4.2 Ester invests R6 000 that he got for her birthday. The bank offers her an interest rate of 5.5% p.a., compounded monthly. After 3 years, she adds R2 000 to her investment and the interest rate changes to 6.5% p.a., compounded bi-annually. After another 2 years, she adds R1 500 to her investment .How much will her investment be worth after 8 years if the interest is 6% p.a., compounded weekly, during the last year? (6)

3.4.3 Thaba invests R25 000 at a bank in an interest rate of 8% p.a., compounded monthly. After 5 years, He withdraws R7 000 to buy a TV-set. What will the balance of his investment be after 8 years if the interest changes to 6.5% p.a., compounded quarterly, for the last 2 years. (4)

3.4.4 Hilda invests R5 million into an account earning interest of 8% per annum,compounded monthly. She withdraws an allowance of R50 000 per month. The first withdrawal is exactly one month after she has deposited the R5 million .How many withdrawals will Hilda be able to make? (6)

3.5 PRACTICE QUESTION D

3.5.1 On the 1^{st} day of January 2019 a school bought a new photocopy machine for R180 000

- The value of the photocopy machine decreases by 25% annually on the reducing balance method.
- When the book value of the photocopy machine is R52 145, the school will replace the photocopy machine.

3.5.1.1 Calculate the book value of the photocopy machine on the 1^{st} day of January 2021. (3)

3.5.1.2 At the beginning of which year will the school have to replace the photocopy machine. (3)

3.5.1.3 The cost of a similar photocopy machine will be R450 000 at the beginning of 2024.The school will use the R52 145 that it will receive from the sale of the old photocopy machine to cover some of the costs of replacing the photocopy machine. The school set up a sinking fund to cover the balance. The fund pays interest at 7.5% per annum ,compounded quarterly .The first deposit was made on the 1^{st} April 2019 and every three months thereafter until 1^{st} January 2024.Calculate the amount that should be deposited every three months to have enough money to replace the photocopy machine on 1^{st} January 2020

(4)

POSSIBLE SOLUTION

3.2.1	$A = P(1+i)^n$ $2x = x(1+\frac{0.08}{2})^{2n}$ $2 = (1.04)^{2n}$ $2n = \log_{1.04} 2$ $2n = 17.67...$ $\therefore n = 8.836....$ ≈ 8.84 years	✓ correct formula ✓ substitution ✓ introduction of log ✓ answer (4)
3.2.2	Loan amount $= \frac{90}{100} \times 450000$ $= 405000$	✓ $\frac{90}{100} \times 450000$ ✓ answer (2)
3.2.3	$1 + i^{eff} = (1+\frac{i^m}{m})^m$ $1 + i^{eff} = (1+\frac{0.08}{12})^{12}$ $1 + i^{eff} = 1.0829995681$ $\therefore i^{eff} = 0.08299950681$ $\therefore i^{eff} = 8.30\%$ $\neq 8.6\%$ **Not correct**	✓ correct formula ✓ substitution ✓ $i = 0.829....$ ✓ conclusion (4)
3.2.4	Given data $n = 6 \times 12 = 72;\ i = \frac{0.095}{12};\ Pv = 405000$ $x = ?$ $Pv = \dfrac{x\left[1-(1+i)^{-n}\right]}{i}$ $405000 = \dfrac{x\left[1-\left(1+\dfrac{0.095}{12}\right)^{-72}\right]}{\dfrac{0.095}{12}}$ $x = R7401.25$	✓ correct formula ✓✓ correct subst. ✓ answer (4)

3.2.5	Given data $x = 8000;\ pv = 405000,\ i = \frac{0.095}{12};\ n = ?$ $Pv = \frac{x\left[1-(1+i)^{-n}\right]}{i}$ $405000 = \frac{8000\left[1-\left(1+\frac{0.095}{12}\right)^{-12n}\right]}{\frac{0.095}{12}}$ $405000 \times \frac{0.095}{12} = 8000\left[1-\left(1+\frac{0.095}{12}\right)^{\to -12n}\right]$ $\left(1+\frac{0.095}{12}\right)^{-12n} = 1-\frac{513}{1280}$ $\left(1+\frac{0.095}{12}\right)^{-12n} = \frac{767}{1280}$ $-12n = \log_{\left(1+\frac{0.095}{12}\right)} \frac{767}{1280}$ $-12n = -64.94565055$ $n = 5{,}41 years$	✓ correct formula ✓ correct subst. ✓ simplification ✓ introduction of log ✓ answer (5)
3.3	**Possible answers**	
3.3.1	Given data $p = 12000;\ n = 12;\ i = \frac{8}{100} = 0.08;\ A = ?$ $A = P(1+i)^n$ $= 2000\left(1+\frac{8}{100}\right)^{12}$ $= R30218.04$	✓ correct subst. ✓ answer (2)
3.3.2	Final gain $= R30218.04 - 12000$ $= R18128.04$	✓ answer (1)
3.3.3	$Pv = \frac{x\left[1-(1+i)^{-n}\right]}{i}$	✓ correct formula ✓✓ correct subst.

	$2000000=\dfrac{x\left[1-\left(1+\frac{0.085}{12}\right)^{-240}\right]}{\frac{0.085}{12}}$ $x=R17356.46$	✓ answer (4)
3.3.4	**Outsanding balance** $=A-FV$ $=P(1+i)^n-\dfrac{x\left[(1+i)^n-1\right]}{i}$ $=2000000\left(1+\frac{0.085}{12}\right)^{12}-\dfrac{17356.46\left[\left(1+\frac{0.085}{12}\right)^{12}\right]}{\frac{0.085}{12}}$ $=2176781.812-2666910.102$ ✓ $=R490128.29$ ✓	✓ correct subst. ✓ simplification ✓ answer (3)
3.3.5	**Capital redeemed** $=2000000-490128.29$ $=1509871.71$ Interest charged $=R4165550.40-1509871.71$ $=R2665678.69$	✓ answer ✓✓ final answer (3)
3.4	**Possible solution C**	
3.4.1	$A=P(1+i)^n$ $=3000000(1+0.005)^4$ ✓ $=R3646518.75$ ✓	✓ Correct formula ✓ subst. ✓ answer (3)
3.4.2	$A=\left[\left[6000\left(1+\frac{0.055}{12}\right)^{3\times12}+2000\right]\left[\left(1+\frac{0.065}{2}\right)^{2\times2}+1500\right]\right]\times$ $\left[\left(1+\frac{0.065}{2}\right)^{2\times2}\left(1+\frac{0.06}{52}\right)^{1\times52}\right]$ =R14253.70 OR	✓ $i=\frac{0.055}{12}$ & $n=36$ ✓ $i=\frac{0.065}{2}$ & $n=4$ ✓ $i=\frac{0.06}{2}$ ✓ subst. $6000\left(1+\frac{0.055}{12}\right)^{3\times12}+2000$

	STEP 1 $A=P\left(1+\frac{i}{12}\right)^{n\times12}$ $=6000\left(1+\frac{0.055}{12}\right)^{3\times12}$ $=R7073.691615$ **STEP 2** $7073.691615+2000=9073.691615$ $A=9073.691615\left(1+\frac{0.065}{2}\right)^{2\times2}$ =R10312.0321 **STEP 3** 10312.0321+1500 =R11812.0321 $A=11812.0321\left(1+\frac{0.065}{2}\right)^{2\times2}$ $A=R13424.09014$ **STEP 4** $A=13424.09014(1+\frac{0.06}{52})^{1\times52}$ $A=R14253.70$	✓ subst. $\left[\left(1+\frac{0.065}{2}\right)^{2\times2}+1500\right]$ ✓ R14253.70 (6) ✓ $6000\left(1+\frac{0.055}{12}\right)^{3\times12}$ ✓ answer ✓ $9073.691615(1+\frac{0.065}{2})$ 2×2 ✓ answer ✓ R13 424.09014 ✓ final answer (6)
3.4.3	$A=\left[25000\left(1+\frac{0.08}{12}\right)^{5\times12}-7000\right]\left(1+\frac{0.08}{12}\right)^{1\times12}\left(1+\frac{0.065}{4}\right)^{2\times4}$ $=R37265.14$ OR STEP 1 $A=P\left(1+\frac{i}{12}\right)^{n\times12}$ $=25000\left(1+\frac{0.08}{12}\right)^{5\times12}$ =R37246.14271 STEP 2 $37246.14271-7000=R30246.14271$ $A=30246.14271\left(1+\frac{0.08}{12}\right)^{1\times12}$ $=32756.55764$ STEP 3	✓ ✓ ✓ ✓ ✓ R37246.14271 ✓ answer ✓ subst. ✓ final answer (4)

	$A = 32756.55764(1+\frac{0.065}{4})^{2\times 4}$ $= R37265.14$	

3.4.4	$Pv = \dfrac{x\left[1-(1+i)^{-n}\right]}{i}$ ✓ $5000000 = \dfrac{50000\left[1-\left(1+\frac{0.08}{12}\right)^{-n}\right]}{\frac{0.08}{12}}$ ✓ $\dfrac{5000000\times\frac{0.08}{12}}{50000} = \left[1-\left(1+\frac{0.08}{12}\right)^{-n}\right]$ ✓ $\left(1+\frac{0.08}{12}\right)^{-n} = 1-\frac{2}{3}$ ✓ $\left(1+\frac{0.08}{12}\right)^{-n} = \frac{1}{3}$ $-n = \log_{\left(1+\frac{0.08}{12}\right)}\frac{1}{3}$ ✓ $-n = -165.3405411$ $\therefore n = 165.340....$ $n = 165$ withdrawals	✓ correct formula ✓ subst. ✓ simplification ✓ simplification ✓ introduce log ✓ $n = 165$
3.5	Possible answers D	
3.5.1	$A = P(1-i)^n$ $= 180000(1-0.25)^2$ ✓ $= R101250$ ✓	✓ correct formula ✓ subst. ✓ answer (3)
3.5.2	$A = P(1-i)^n$ $52145 = 180000(1-0.25)^n$ ✓ $\dfrac{52145}{180000} = (0.75)^n$ $n = \log_{\frac{52145}{180000}} 0.75$ ✓ $n = 4$ ✓ The photocopy machine will be replaced at the beginning of 2024	✓ correct formula ✓ subst. ✓ introduction of log ✓ $n = 4$ (4)
3.5.3	$R180000 - R52145 = R127855$ $Fv = \dfrac{x\left[(1+i)^n - 1\right]}{i}$	✓ $R127855$ ✓ correct formula

	$127855 = \frac{x\left[\left(1+\frac{0.075}{4}\right)^{4\times 4}-1\right]}{\frac{0.075}{4}}$ ✓ $x = \frac{76713}{32} \div 18.45941$ $x = R129.87$ ✓	✓ subst. ✓ answer. (4)

Glossary

1.1 Profit

Profit can be defined as the monetary gain made from a transaction. In consumer arithmetic, a profit is made if the selling price of a product is greater than the cost price (the price at which the product was bought) of the product.

1.2 Percentages (%)

Percentage is a representation of a fraction of the whole sample.

A percentage expresses a part of 100 $\left(e.g.\ x\% = \frac{x}{100}\right)$.

1.3 Percentage profit

The percentage profit made from the transaction is found by using the formula:

1.4 Loss

A loss is made from a transaction if the selling price of a product is less than the cost price.

Therefore, Loss = Cost price – Selling price

1.5 Percentage loss

The percentage loss made from the transaction is found by using the formula:

1.6 Deposits

A sum of money paid up front before a loan is granted.

1.7 Investment

Money you put in the bank in the hope that it will grow.

1.8 Unit Trust

Shares

1.9 Annual compound growth rate

The interest rate per year

1.10 Depreciation

The **decrease** in value of an asset over a period of time.

When equipment loses value over time, we say that the equipment is depreciating in value. For example, the moment a new car is driven out of the garage/dealership, its value depreciates substantially. Obviously, due to wear and tear, the car will lose its value over time.

1.11 Simple decrease or (Straight line) depreciation

Simple decrease is also known as straight line decrease

This is depreciation based on the original value and never changes. i.e., the amount by which the value of an asset decreases every year is a constant percentage of the original value.

With linear depreciation, equipment is depreciated by a percentage of its original value. It can be represented by a "linear" function.

It works in the same way as simple interest, but the value decreases rather than increases as with simple interest.

$A = P(1 - in)$ where,

A = end amount, P = initial amount, n = number of periods of the decrease, e.g., number of years and I = rate of decrease divided by 100;

1.12 Inflation

Inflation is the steady compounded increase in prices over time throughout the economy.

The effect of inflation is to erode the buying power of money over time.

The rate of increase worked out by compound interest annually;

1.13 Hire Purchase Agreements

A hire-purchase agreement (HP) is a short-term loan.

1.14 Per annum

Annually

1.15 Trade in

The money you are granted after depreciation

1.16 Interest

Interest is the fee paid by a borrower to a lender for the use of borrowed money; the fee is usually expressed as an annual percentage of the amount borrowed.

1.17 Simple Interest

Simple interest describes situations in which money is borrowed or invested for a period of time and interest is calculated once only and paid at the end of the period.

1.18 Compound interest

Compound interest is used to describe situations in which money is borrowed or invested for a period of time and interest is calculated on the balance at intervals throughout the period and added to the balance.

1.19 Nominal interest rate

A **nominal rate** is quoted as an annual rate, without taking into consideration the effect of different compounding periods, which are shorter than the annual period.

1.20 Effective Rate

Effective annual interest rates are equivalent annual rates that yield the same accumulated amount as rates with different compounding periods (monthly, quarterly, half-yearly, daily).

1.21 Annuities

The term **"Annuity"** is used in Financial Mathematics to refer to any terminating sequence of **regular fixed payments** over a specified period of time. If payments are not regular **(irregular)** periods, we are not working with an annuity.

1.22 Present Value Annuity

"Present value annuity" is a loan.

1.23 3.5.1.3 Future Value Annuity

"Future value annuity" is a savings plan for the future.
period;

1.24 Loan

Loan is when money is borrowed usually from an institution and repaid over usually a fixed interval in regular instalments;

1.25 Deferred Annuity

In many situations, the repayment of a loan begins three months after the granting of the loan has been made. However, circumstances arise where the repayment of the loan is deferred for an agreement of time. Then you would have to add three months interest on the loan amount to the principal amount.

1.26 Scrap (Depreciated) Value

The value after depreciation.

1.27 Book Value

It is the value of equipment at a particular time after depreciation has been taken into account.

1.28 Sinking Fund

A **Sinking Fund** is an investment that is made in the form of regular payment for the express purpose of meeting an expected cost,

1.29 Laws

The interest period must be same as the repayment intervals. That is, if the interest is quoted as quarterly, and the repayments of investments are monthly, the interest must be converted to a monthly interest rate.

1.30 Bond repayment

A **bond** repayment is when you have borrowed some money (to buy a house or building) at a given interest rate and you then make regular payments to pay off the original amount of money borrowed as well as any interest that has arisen. (The bond is

The term **amortization** is also applied to the situation where a regular payment is made to pay off a loan with interest. The loan is then said to be amortized.

3. Study and Examination Tips

- Learn the correct use of calculators in order to prevent step by step answers.
- Rounding should be correct to two decimal places.
- Re-read with understanding should be practice with limited given to avoid generalization.
- Encourage learners to use exponential laws and understand how to make the subject of the formula, which involves previous grades, in order to avoid using the incorrect formula.
- Basic algebraic rules should be appropriately taught.
- Correct interpretation of financial language need to be taken into consideration.
- Teachers should use the correct language/concepts in class and in assessment.

- Use a wide of resources to enhance the skills and knowledge in the classroom.

- NB: Please read your question in the examination attentively before answering them.

Reference

- Mathematics CAPS document
- Mathematics Examination Guideline
- Previous NSC and Preliminary Examination papers 2016 -2020
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